

**Capturing Volatility Smiles with a Perpetual Leverage
Model, and its Implications to Fund Overlay Designs**

by

Min Chen

A dissertation submitted to The Johns Hopkins University in conformity with the
requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

April, 2016

© Min Chen 2016

All rights reserved

Abstract

Structural modeling of leveraged firms treats a firm's equity as a derivative whose underlying instrument is the firm's asset. For example, Merton (1974) modeled the firm's equity as a vanilla call option, and Leland (1994) modeled equity as a perpetual option. Under this assumption, the equity option of a leveraged firm is then a compound option, as demonstrated by Geske (1979) and by Toft and Prucyk (1997). The compound option assumption is a powerful tool to for explaining the volatility smile and skew observed in the equity option market, as demonstrated in Toft and Prucyk (1997), Hull et al. (2004a) and Chen and Kou (2009). However efforts to understand the smile and skew observations through structural modeling have been limited.

This thesis further explores the explicit representation of volatility smiles/skews through structural modeling, achieving a better replication of the market with prac-

ABSTRACT

tical modeling and calibration strategies. The equity of a firm is modeled as a perpetual option, but in a more general format, compared to existing publications following Leland. Asymmetry is introduced into asset return distributions through a constant elasticity of variance (CEV) process, so that the model achieves better agreements to skews, smiles, and when the leverage is insignificant. The choice of CEV asset stochastic ensures enough model flexibility to produce various shapes of volatility skew and smile. It also retains the mathematical tractability allowing the calibration of the compound option to the vanilla market to remain practical. Lastly, the model remains moderately parameterized so that the calibration is still meaningful. This calibration produces leverage metrics potentially helpful to fundamental and credit analysis.

The equity-asset relation under CEV asset assumption is modeled through a free-boundary differential equation, which reflects the financial aspects of a limited liability firm. The equity value is solved from this free-boundary problem as a closed-form relation with respect to the firm's asset dynamic and its nominal liability. This closed-form representation is then embedded into the equity option pricing model, simplifying the compound option pricing problem as a more approachable barrier option pricing (first seen in Toft and Prucyk (1997)). A practical Monte-Carlo based

ABSTRACT

fitting strategy is proposed, so that the model can be tested on a larger set of candidates within a reasonable amount of computation time.

Empirical tests demonstrate the capabilities of this model to produce both volatility smile and skew, to accommodate the volatility skew observed on very low-leverage firms, and to produce credit quality measures that are more consistent to the credit default swap (CDS) market. Distribution analysis on S&P-100 and NASDAQ-100 candidates generates distinct leverage and volatility distributions between the two index pools that are consistent with the component characteristics of each pool.

Some desirable features of the perpetual structural model also inspired additional discoveries in retail fund management. A perpetual American put option replication strategy is provided as an investment protection, whose benefits, including extreme loss prevention and path-independency, are also illustrated.

Advisor & Primary Reader: Daniel Q. Naiman

Secondary Reader: Hélyette Geman

Acknowledgments

I am grateful to Professor Daniel Naiman for advising me through this research. I would like to thank Professor Hélyette Geman for a series of thoughtful conversations that greatly improved this thesis. I also wish to thank Professor Maxim Bichuch for reading my thesis and for all the suggestions. Professor Steve Hanke and Professor Jonathan Wright kindly shared their opinions and were very helpful as well. The advice of Professor David Audley and Professor Shih-Ping Han was sincerely appreciated in the early stages of this research. I wish to thank my Ph.D. colleague Yichen Qin for sharing his insights on statistical analysis. I am grateful to the management of Ohio National Financial Services for their many forms of support as I pursued this thesis.

Lastly, I wish to thank my parents for all their love and support.

Dedication

To my parents, and Ballspielverein Borussia Dortmund...

Contents

Abstract	ii
Acknowledgments	v
List of Tables	x
List of Figures	xi
1 Introduction	1
1.1 Motivation of Research	1
1.2 Literature Review	4
1.3 Overview and Contributions	17
1.4 Organization	20
2 Theoretical Development of the CEV Leverage Model	21
2.1 Asset-Equity Relation and the Key Result	25

CONTENTS

2.2	Equity Option Pricing and Market Calibration	35
2.3	Derivation of the Key Result	41
2.3.1	The Free-Boundary Problem Formulation	41
2.3.2	Solving the Free-Boundary Problem	47
2.4	Implementation Concerns of the Monte Carlo Based Calibration . . .	51
3	Structural Leverage Model Empirical Studies	56
3.1	Reproducing and calibrating to volatility smile	58
3.2	Volatility skew of low leverage firms	61
3.3	Rank consistency analysis between expected default loss and CDS spread	63
3.4	Additional Challenges in Cross-sectional Study of CEV-based Struc- tural Model	71
3.5	Cross-sectional Parameter Distribution Analysis	73
4	An Investment Protection Overlay based on Perpetual American Put Option	81
4.1	Protection Overlay Introduction	81
4.2	Existing strategies	84
4.3	PAP-based protection overlay	87
4.4	Impact of protection overlay on hedging	91

CONTENTS

5	Conclusions and Future Work	104
5.1	Conclusions	104
5.2	Future Work	106
A	Notations	108
B	Additional Literatures	110
C	Historical Maximum Likelihood Estimation of Heston Model	117
	Vita	134

List of Tables

4.1	PAP protection's impact on annual returns and maximum losses . . .	91
4.2	Illustrating the hedge strategy simulation	96

List of Figures

2.1	Illustration of the free-boundary differential equation and the boundary conditions	46
3.1	Volatility skews and smiles under different elasticity when adjusted volatility is 0.3 and leverage is 0.2	59
3.2	Calibration to YAHOO Inc. volatility smile observed on Jan.17, 2014	60
3.3	AAPL volatility skew calibrated under both CEV and GBM asset dynamic	62
3.4	Evolution of the probability of default under the framework of Kolmogorov backward equation.	65
3.5	Relationship between (a) CEV EDL (b) JMR PD and CDS, NYSE:AIG	69
3.6	Relationship between (a) CEV EDL (b) JMR PD and CDS, NYSE:SPG	70
3.7	Kernel smoothing function estimation of firm volatility under (a) CEV (b) JMR implementation	76
3.8	Kernel smoothing function estimation of implied firm leverage under (a) CEV (b) JMR implementation	78
3.9	Kernel smoothing function estimation of asset return elasticity (α) for CEV implementation	80
4.1	The performance of volatility controlled fund under progressive market crash, with a more desirable alternative.	85
4.2	An example of the price relation between an unprotected fund and a PAP-protected fund: \$100 strike, 2% interest rate and 20% volatility	88
4.3	Historical performance of perpetual American put overlay in 2002 and 2008	90

LIST OF FIGURES

4.4	One example of the hedge cost frontier. Each point on the frontier indicates that to achieve the desired hedging goal, a put purchase with the combination of fractional strike ($\frac{\text{Put Strike}}{\text{Initial Value}}$) and fractional shares ($\frac{\text{Share of Puts Purchased}}{\text{Shares of Investments}}$) is required.	93
4.5	The hedge cost frontiers for both PAP (solid) and volatility targeting (dashed). Each point on the frontier indicates that to achieve the desired hedging goal, a put purchase with the combination of fractional strike ($\frac{\text{Put Strike}}{\text{Initial Value}}$) and fractional shares ($\frac{\text{Share of Puts Purchased}}{\text{Shares of Investments}}$) is required.	99
4.6	Relation of Log returns between unprotected and PAP protected funds at the end of insurance policy over all scenarios	100
4.7	The relation of maximum loss of two strategies over all scenarios . . .	101

Chapter 1

Introduction

1.1 Motivation of Research

Not long after the introduction of geometric Brownian motion (GBM, by Samuelson (1965)) modeling of stocks and the Black and Scholes (1973) option pricing formula, the volatility smile and skew have quickly become commonly observed in the stock option market. One pervasive explanation for this phenomenon is the leverage effect, which suggests that a falling equity value will lead to an increase of leverage, because the liability obligations will remain steady regardless of the firm's performance, whereas the asset and equity will both decrease. In this situation, even if the asset volatility is assumed to be constant for modeling convenience, the equity

CHAPTER 1. INTRODUCTION

volatility will still grow due to the increasing leverage. Additionally, the falling equity value is usually accompanied by increased asset volatility, which, when coupled with the leverage increase, significantly raises the market expectation of tail event likelihoods, and therefore creates an asymmetric and fat-tailed risk-neutral distribution for the equity returns. The change of leverage is considered to be one of the major contributors to the divergence of the risk neutral asset return distribution from being Gaussian, and as a result, the phenomenon of volatility skew and smile.

To address the non-Gaussian property of the equity return distribution and to achieve a more efficient market calibration, various extensions to the Samuelson's geometric Brownian motion equity dynamic have been proposed. These models include the constant elasticity of variance (CEV) model by Cox (1996) (originally in 1975), the local volatility model by Dupire (1994), the stochastic volatility model by Heston (1993), the jump diffusion model by Merton (1976), and pure jump Levy process models (e.g. the CGMY process by Carr et al. (2002)).

By introducing various kinds of non-Gaussian stochastic processes, many of these models can efficiently replicate and calibrate to the implied volatility structure with a few parameters, as opposed to calibrating one single market with multiple param-

CHAPTER 1. INTRODUCTION

eter sets as seen in the practical applications of Black-Scholes model. However, most of these models treat the equity dynamic as a black box and are therefore exempt from addressing explicitly the structural reasons why equity returns follow a certain dynamic.

Since it is widely agreed that leverage is the key contributor to the non-Gaussian risk-neutral equity return distribution derived from the market, the leverage information must have been incorporated into the implied volatility structure. An alternative strategy which explicitly models the leverage and reflects it in the equity option prices has the potential to extract leverage information from the implied volatility structure. Such a strategy is highly desirable because the information inferred from the market can be useful in many applications, including assisting fundamental and credit analysis.

This study attempts to establish a stronger link between equity option pricing and corporate leverage modeling. By setting up a CEV asset model in which the leverage has a direct impact on equity option prices, the model can be calibrated to the implied volatility structure, so that the leverage can be inferred from the equity option market. Empirical study suggests that this implied leverage is a more desirable

CHAPTER 1. INTRODUCTION

measurement for corporate liability than values extracted from financial statements. This empirical study also highlights the necessity of introducing asymmetric asset returns alongside the leverage in order to obtain a reasonable fit to the implied volatility structure.

1.2 Literature Review

Equity holders leverage their business by borrowing and therefore putting themselves under liabilities. They can benefit from the growth of the business as long as the liability obligations are fulfilled. However, the limited liability nature of modern firms gives equity holders the right to declare default at any time, insulating them from any further claims when keeping the business running is no longer in their best interest. In the event of a default, debt holders will take any of the firm's remaining value, even though the default is completely the equity holders' decision.

Equity holders tend to make optimal default decisions, and thus embed “optionality” into the equity value of the firm. For this reason, equity value depends nonlinearly on the asset and the liability of the issuing firm. This is to say that a stock is actually an option on the firm's asset, and that stock options are therefore compound options whose original source of randomness is the asset dynamic and the

CHAPTER 1. INTRODUCTION

leverage.

Even if the asset return distribution is sometimes assumed to be symmetrical for modeling convenience, the embedded optionality of equity introduces nonlinearity between asset return and equity return. This nonlinearity creates asymmetry in the equity return distribution and can play a key role in reproducing the implied volatility skew/smile observed in the stock option market (see Section 3.1 for discussions of how leverage can affect volatility skew). For this reason, a model which quantifies the impact of this optionality on the equity return asymmetry is highly desirable for understanding this complex dynamic. Such a model will build a strong link between leverage and implied volatility structures.

Similar to the rational pricing of equity options, the quantitative modeling of corporate leverage is not an unexplored area. Even before Merton (1974), Black and Scholes (1973) mentioned the option property of common stocks in their revolutionary 1973 option pricing paper. Merton formalized this logic and initialized a whole area of structural modeling by viewing corporate equity as a vanilla call option on the asset. Merton's revolutionary work reminds the financial world of an oft-ignored fact: the equity of a leveraged firm, which has long been considered a primitive se-

CHAPTER 1. INTRODUCTION

curity, is instead a derivative.

There is a tremendous amount of research following the Merton (1974) model. Before further developing the desired model for this research, seven existing studies closely related to this work are reviewed here in detail. These seven papers cover the building blocks of this thesis, including the option modeling of firms' equity values, the reversed implementation of the structural models, the perpetual option based leverage modeling, the construction of equity volatility skews/smiles, the calibration of structural models to the volatility skews/smiles, and the departure from the Gaussian assumption on the firms' asset returns. Reviewing previous research on this topic will help readers to appreciate the historical path that lead to the model presented in this thesis. These papers are Merton (1974), Jones et al. (1984), Geske (1979), Hull et al. (2004a), Leland (1994), Toft and Prucyk (1997), and Chen and Kou (2009).

Merton is considered to be the pioneer in this area because he was the first to explicitly model the equity of a limited liability firm as a derivative whose underlying instrument is the asset value of the firm. Merton assumed that the firm's asset value

CHAPTER 1. INTRODUCTION

is driven by a GBM asset stochastic:

$$dV = rVdt + \sigma_V V d\tilde{W} \quad (1.1)$$

where \tilde{W} is a Brownian motion defined on probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, Ω is the outcome set of \tilde{W} , \mathcal{F} is the filtration where \tilde{W} is measurable, and \mathbb{Q} is the risk-neutral probability measure. Merton also simplified corporate liability into a single zero-coupon bond maturing at time T with face value D . Under these assumptions, the equity of a firm is a European call option on the asset value of the firm with maturity T and strike D . At the maturity of debt, equity holders will either continue to operate the firm by exercising the European call option for the marginal benefit $V(T) - D$ when $V(T) > D$, or elect to be exempt from the liabilities by giving up the right to exercise this European call option when $V(T) < D$. The value of equity can then be represented by the Black-Scholes' formula:

$$E(V, t) = N(d_+)V - N(d_-)De^{-r(T-t)} \quad (1.2)$$

where

$$d_{\pm} = \frac{1}{\sigma_V \sqrt{(T-t)}} \left[\ln \left(\frac{V}{D} \right) + \left(r \pm \frac{\sigma_V^2}{2} (T-t) \right) \right] \quad (1.3)$$

and $N(\cdot)$ is the cumulative distribution function for standard normal distributions.

CHAPTER 1. INTRODUCTION

Several questions remained unanswered in Merton (1974) research. Specifically, the estimation of each of the four inputs, V , σ_V , D and T , is a significant challenge. One can argue that D and T are liability related variables, which can be reliably estimated from financial statements. However, the lack of V and σ_V inputs still poses problems because of the volatile nature of the asset. A marked-to-market estimation is therefore more preferable to financial statement reading.

On the contrary, the output of (1.2) is directly readable from the stock market. Therefore, a quick remedy for the missing asset dynamic parameters is to adjust V and σ_V so that the equity price estimated by (1.2) agrees with the stock price. However, it is still not fully robust to estimate two parameters from one equation, and it was Jones et al. (1984) (JMR hereafter) that closed this gap. Applying the quadratic variance argument of Brownian motion, these two conclusions follow naturally (inheriting notations and definitions from Merton's model):

$$dVdV = r^2V^2dt + 2r\sigma_VV^2dt\tilde{W} + \sigma_V^2V^2d\tilde{W}d\tilde{W} = \sigma_V^2V^2dt \quad (1.4)$$

CHAPTER 1. INTRODUCTION

and

$$dE = \frac{\partial E}{\partial t}dt + \frac{\partial E}{\partial V}dV + \frac{1}{2}\frac{\partial^2 E}{\partial V^2}dVdV \quad (1.5)$$

Substituting (1.4) into (1.5) produces the basic relation:

$$dE = \left[\frac{\partial E}{\partial t} + rV\frac{\partial E}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 E}{\partial V^2} \right] dt + \sigma_V V \frac{\partial E}{\partial V} d\tilde{W} \quad (1.6)$$

Equating (1.6) with the general drift-diffusion equation

$$dE = rEdt + \sigma_E E d\tilde{W} \quad (1.7)$$

leads to Black-scholes' PDE (through equating the drift term) as well as the instantaneous volatility equation (through equating the diffusion term)

$$\sigma_E E = \sigma_V V \frac{\partial E}{\partial V} \quad (1.8)$$

where $\frac{\partial E}{\partial V}$ is simply the Black-Scholes' Delta:

$$\frac{\partial E}{\partial V} = N(d_+) - 1 \quad (1.9)$$

JMR were the first to suggest that (1.2) and (1.8) can be jointly solved to provide

CHAPTER 1. INTRODUCTION

a robust estimation of both $V(0)$ and σ_V , once D and T are specified by the liability structure and $E(0)$ and σ_E are observed from the stock market.

JMR's implementation made several important contributions to this area of study. First of all, they provide a robust method for estimating asset return volatility, which is a latent variable but may have interesting implications for fundamental and credit analysts. Second, the JMR implementation changed the flavor of Merton's model, in that it evolved the model from an economic model to a mathematical finance model. Instead of describing the default dynamic in a theoretical context, the model now has the capability to process market data and produce advanced metrics. In the context of this thesis, the most important contribution of the JMR implementation is that it linked the structural model to equity volatility, which inspired later researchers to incorporate stock options into this area of research. The JMR implementation also made a significant impact in the industry: models and even companies (e.g. Crouhy et al. (2000)) specialized in producing credit metrics out of equity market data emerged based on JMR's theoretical contributions.

Nevertheless, several implementation details in JMR's work still needed to be addressed. For example, the estimation of D and T remains unresolved, since the

CHAPTER 1. INTRODUCTION

complex liability structure cannot be easily and systematically mapped onto one single zero-coupon bond. Also, the forward-looking equity volatility cannot be easily predicted. One straightforward solution would be to use the historical realized volatility, but historical measure may not lead to a good prediction of future realizations. From this standpoint, the option-implied volatility would be a better choice. At the time of JMR's publication, implied volatility skew/smile was still a relatively new concept that was not a concern for most analysts, so at-the-money implied volatility was a very acceptable input. However, in later years, analysts have become more prudent about the phenomenon of volatility skew and smile, which has led to a commonly held belief that complex volatility structures are closely related to the leverage or the capital structure. Two notable papers representing different directions of research in structural modeling of equity volatility skew are those by Hull et al. (2004a) and Toft and Prucyk (1997).

In 1977 and 1979, Geske published two papers extending Merton's pioneering work. In these two papers, Geske presented the innovative idea that equity options are actually compound options, which is an option written on another option given that Merton has revealed the option nature of firms' equity. With the successful pricing of the compound options in 1979, Geske suggested that a put option with

CHAPTER 1. INTRODUCTION

strike K and expiration τ (assume $\tau < T$ where T is the expiration of the zero coupon bond in Merton's model) is priced as:

$$p = De^{-rT}M\left(-a_2, d_2; -\sqrt{\frac{\tau}{T}}\right) - V_0M\left(-a_1, d_1; -\sqrt{\frac{\tau}{T}}\right) + Ke^{-r\tau}N(-a_2) \quad (1.10)$$

where $M(\cdot, \cdot, \cdot)$ is the bivariate normal distribution (the third argument is the correlation coefficient), $N(\cdot)$ is the standard Gaussian distribution,

$$a_1 = \frac{\ln(V_0/V_\tau^*e^{-r\tau})}{\sigma_V\sqrt{\tau}} + \frac{1}{2}\sigma_V\sqrt{\tau} \quad (1.11)$$

and $a_2 = a_1 - \frac{1}{2}\sigma_V\sqrt{\tau}$,

$$d_1 = \frac{-\ln(D/V_0)}{\sigma_V\sqrt{T}} + \frac{1}{2}\sigma_V\sqrt{T} \quad (1.12)$$

and $d_2 = d_1 - \sigma_A\sqrt{T}$. V_τ^* is the level of asset such that at τ , the expiration of option:

$$E(V_\tau^*) = K \quad (1.13)$$

Note that in Geske (1979)'s original work, a call option was priced. However, this is equivalent to pricing a put option, and for the convenience of introducing the work by Hull et al. (2004a), the put pricing formula is shown here instead of the call pricing. All notations are adjusted to be consistent with Hull et al. (2004a).

CHAPTER 1. INTRODUCTION

Geske also revealed that Black and Scholes (1973)’s result is a special case of his compound model when the underlying firm has zero leverage. He also noted that when the “elasticity of the stock price with respect to the value of the firm is assumed to equal a power function of” the stock price, “the compound option model reduces to a form of the constant elasticity of variance models”. This discovery implicitly revealed that the structural modeling can have important implications to volatility skews because the CEV model is just one type of alternative model designed to explain skews. However, more explicit modeling of volatility skew following Geske’s compound structure was not reported until Hull et al. (2004a).

Based on Geske (1979), Hull et al. (2004a) explored the impact of leverage on volatility skew. Based on (1.10), they proposed a strategy of using two different implied volatilities to infer leverage and asset volatility. Instead of calibrating to the entire volatility skew, Hull et al. used a linear approximation to the skew based on the selection of a pair of strikes. The GBM asset dynamic assumption and Merton (1974)’s single zero-coupon bond assumption remained in this research. Instead of depending on the reading of financial statements, Hull et al. mapped the complex liability structure onto one zero-coupon bond with a given maturity by the market

CHAPTER 1. INTRODUCTION

assessment reflected in the volatility skew.

This was a breakthrough in structural model research, because, for the first time, the liability fell out of the set of inputs and became a parameter to be implied by the options market. This created a wide new area to be explored by researchers. That being said, the maturity of such a zero-coupon bond (i.e. the liability) still needs to be manually specified. Also, in the empirical portion of this paper, the “50-delta and 25-delta implied volatilities” were selected as the pair of strikes for liquidity benefits. However, the impact of selecting different maturities or strike pairs on the implied smile/skew was not discussed.

The requirement of specifications to liability maturity is a key restriction under the Merton-Geske framework. When modeling firm equity with a vanilla option, the maturity of liability inevitably requires an input, which cannot be determined with confidence. This restriction leads the research attention to another set of papers represented by Leland (1994) and Toft and Prucyk (1997) using a perpetual liability assumption. This restriction also leads to one of the model choices made in this research, discussed in Section 2.1.

CHAPTER 1. INTRODUCTION

Leland (1994) was among the first to walk away from Merton's European option framework. By assuming a GBM asset dynamic and constant-coupon-only liability structure, Leland derived an alternative structural model based on perpetual American option theory (ignoring the dividend assumption):

$$E(V) = V - C/r + [C/r - V_B][V/V_B]^{-2r/\sigma^2} \quad (1.14)$$

where C is the constant and perpetual annual coupon payment, and $V_B = C/(r + \frac{1}{2}\sigma^2)$.

Based on Leland (1994), Toft and Prucyk (1997) proposed a compound option framework to price vanilla options on a firm's equity: that is, since the equity of a firm is an option on the asset, the stock option of a firm is then a compound option written on this perpetual option. They implemented techniques seen in barrier option pricing to achieve an analytical pricing of a vanilla call of a firm's equity as:

$$\begin{aligned} \text{CALL} = & V e^{-\delta T} \left[N(y^* + \sigma\sqrt{T}) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2+2} N\left(y^* + \sigma\sqrt{T} + \left(\frac{2b}{\sigma\sqrt{T}}\right)\right) \right] \\ & + B \left(\frac{V_b}{V}\right)^x \left[N(y^* - x\sigma\sqrt{T}) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2-2x} N\left(y^* - x\sigma\sqrt{T} + \left(\frac{2b}{\sigma\sqrt{T}}\right)\right) \right] \\ & - (A + K)e^{-rT} \left[N(y^*) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2} N\left(y^* + \left(\frac{2b}{\sigma\sqrt{T}}\right)\right) \right] \quad (1.15) \end{aligned}$$

CHAPTER 1. INTRODUCTION

where $N(\cdot)$ is the standard Gaussian distribution, δ is the dividend yield, $\mu = r - \delta - \frac{1}{2}\sigma^2$, $V_b = \frac{Cx}{r(x+1)}$, $b = \ln(V_b/V)$, $x = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$, $A = \frac{C}{r}$, $B = (A - V_b)$, K is the option strike, C is the constant and perpetual annual coupon payment, y^* can be obtained by solving:

$$K = V \exp[(r - \delta - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}(-y)] - A + B \left(\frac{V_b}{V}\right)^x \exp[-x(r - \delta - \frac{1}{2}\sigma^2)T - x\sigma\sqrt{T}(-y)] \quad (1.16)$$

From this equation they theoretically demonstrated that “the volatility skew is negatively related to leverage” and verified the conclusion by regression-based statistical tests on empirical data:

$$\text{Volatility Skew} = a + b\text{Leverage} + \epsilon \quad (1.17)$$

No attempt was made to calibrate the firm asset stochastic to the actual implied volatility skew/smile observed in the market, possibly because the symmetrical asset return assumption remained throughout the study. As seen later in Section 3.2, this assumption can sometimes pose significant challenges to the calibration of the market implied volatility structures.

CHAPTER 1. INTRODUCTION

Chen and Kou (2009) took a significant step away from the GBM asset return assumption in the structural modeling of volatility skew. They extended Leland’s model by introducing two-sided jumps into the GBM asset dynamic. By fine-tuning the heavily parameterized model, Chen and Kou were able to produce “a variety of shapes for the implied volatility of equity options”. Similar to Toft and Prucyk (1997), no strategy was outlined for selecting the model parameters to achieve agreements with volatility skew observed in the market.

Along with these seven highly relevant papers, there are many other studies published in the area of structural modeling of leverage firms. For reference, they are listed in chronological order in Appendix B.

1.3 Overview and Contributions

In this research, a perpetual option framework similar to Leland (1994) is adopted. This avoids the specification of debt maturity, which is not easily estimated with the available data. Similar but still different from Leland’s assumption of a constant coupon payment, the structure of nominal liability is summarized by one constant variable: the perpetual nominal liability level. This leads to a constant default barrier, which later on, combined with the closed-form solution of the equity-asset rela-

CHAPTER 1. INTRODUCTION

tion, helps to simplify the compound option into a more approachable barrier option.

Different from all previous work, the asset dynamic is modeled by a CEV process, which creates a good balance between asset return asymmetry and mathematical tractability. Using a CEV process produces a volatility structure that is flexible enough to match many different kinds of observations in the market. In addition, the relation between asset and equity value remains expressible in closed-form. This is crucial because it significantly simplifies the stock option pricing later on so that the market calibration becomes practical.

The entire volatility structure, together with current equity value, is used to imply the firm asset dynamic as well as the leverage. In short, when the proper parameters are chosen, the model should reproduce the current market price of equity as well as volatility skew or smile.

This model takes into account the entire volatility structure so that no manual selection of strikes is necessary. The calibrated parameters, related to asset volatility, leverage and return asymmetry, reveal critical properties of the firm's fundamentals. The model also resolves a paradox that GBM-based structural models

CHAPTER 1. INTRODUCTION

cannot explain: when the leverage is insignificant, any GBM-based leverage model will produce a very flat volatility structure, which is inconsistent with the market. The CEV-based structural model can produce volatility skews even in the absence of leverage, simply because of the asymmetric asset return.

Another major contribution of this model is that it can accommodate not only the volatility skew but also the volatility smile. Volatility smile is observed in the stock option market with non-trivial probability. If the asset dynamic is modeled as a GBM, the leveraged equity model either produces a flat volatility structure when there is no substantial leverage or a volatility skew when the leverage is significant. Introducing complicated asset dynamics may help to produce richer volatility structures but doing so makes it extremely unstable to calibrate to the market. The CEV asset dynamic provides a balance between the diversity of volatility structure it can accommodate and the mathematical tractability, which are both highly desirable features when calibrating to the market.

The implications of this research are later extended beyond the corporate capital structure analysis. The structural modeling of the corporate liability leads to the decomposition of firm equity value into three components, namely the nominal

CHAPTER 1. INTRODUCTION

liability, asset and a put option, which provides the optionality to equity holders for declaring default. The combination of the underlying asset and the put option forms a systematic fund protection strategy that might be appealing to retail market participants such as mutual fund managers and variable annuity providers. Not only it produces a desirable risk profile which prevents extreme investment losses, it is also fully systematic and thus predictable. For these two reasons it gives greater confidence to hedgers who manage the risk of investments involving products backed by these protected funds.

1.4 Organization

The rest of this thesis is organized as follows: a new CEV-based model allowing for an asymmetric asset dynamic is proposed in Chapter 2. With the closed-form solution to this model, a calibration strategy integrating both the stock and stock option markets is also presented. Empirical study in Chapter 3 shows the model's capability to produce not only the volatility skew but also the smile, and at the same time explains why the volatility structure can still be non-flat even in the absence of significant leverage. The calibrated parameters characterize crucial financial properties of the leveraged firms, and the credit quality measure shows better consistency

CHAPTER 1. INTRODUCTION

to Credit Default Swap (CDS) spread movements. Chapter 4 presents a systematic fund protection strategy inspired by the perpetual option modeling of leveraged firms. Some initial studies demonstrate the desirable features of such a strategy from the standpoints of individual investors and variable annuity hedgers.

Chapter 2

Theoretical Development of the CEV Leverage Model

This chapter presents the theoretical foundation as well as the detailed derivations of the CEV leverage model. The asset dynamics are assumed to follow a constant elasticity of variance (CEV) process and the “nominal liability” (defined later) is assumed to hold at a perpetual and constant level (as a generalization to Leland (1994)). Default is triggered when the asset value hits a critical threshold, which is optimally determined to maximize the benefit of equity holders (commonly known as endogenous default). Under these three assumptions, the equity-asset structure can be modeled as a perpetual derivative represented by a free-boundary differen-

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

tial equation. This equation and its boundary conditions are solved analytically to obtain a closed-form solution of the firm's asset. This analytic solution is then used to simplify the compound option into a barrier option, so that they can be priced through Monte Carlo simulations. Both the CEV asset process and the nominal liability level are calibrated to the stock price and a series of stock options through the Monte Carlo vanilla option pricer (by tuning four parameters including the asset value, asset volatility, elasticity and constant debt level) . The implementation issues are also discussed to ensure the feasibility of such a calibration.

The choice of CEV process finds a good balance between a realistic asset dynamic and a mathematically tractable model. On one hand it provides the necessary degrees of freedom to accommodate various shape of implied volatility smiles/skews (further discussions seen in Section 3.1 in the next chapter), and on the other hand it keeps the equity-asset relation in a moderately parameterized closed form. Such a closed-form solution is critical to the further development of this study, in that it enables the simplification of the compound option into a more approachable barrier option (first seen in Toft and Prucyk (1997)). Compared to an alternative attempt made by Chen and Kou (2009), the CEV model is not as heavily parameterized, and therefore the market calibration becomes more meaningful. With the generalized

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

asset dynamic and consequently the more flexible implied volatility structure, the entire market volatility observation can be accommodated when fitting the model parameters to the stock option market, and therefore avoiding the manual selection of a few strikes (as seen in Hull et al. (2004a)). This market fitting enables the discovery of leverage information solely from the market prices.

The constant nominal liability assumption generalizes the perpetual annuity coupon liability adopted by Leland (1994). It does not require the specification of the liability maturity (seen in, e.g., Jones et al. (1984) and Hull et al. (2004a)), which is not a well-defined concept given the complexity of capital structures of the modern firms. This improvement, together with the compatibility to the entire volatility smile/skew enabled by the CEV process, reduces the need for judgmental processing of financial reports when calibrating the structural model to the market. Intuitively, the nominal liability value has a strong influence to the low-strike end of the implied volatility structure, and the CEV elasticity has a stronger influence to the high strike end. Therefore, both the liability and the elasticity can be implied once a volatility smile/skew is supplied from the market. More illustrations are provided in Section 3.1 of the empirical study chapter (Chapter 3). The mathematical details of calibration is provided in Section 2.2.

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

Due to the structural complexity of this study, this chapter follows a slightly non-standard layout. To keep the discussion focused on the bigger picture, some mathematical details are postponed to Section 2.3. In Section 2.1 the key result, namely, the asset-equity relation based on the CEV leverage model, is first presented without theoretical justification. Next, the vanilla equity option pricing is illustrated as a necessary building block for the introduction of market calibration in Section 2.2. The market calibration procedure is a backward implementation (commonly seen in mathematical finance) of the method developed in 2.1, which tackles the difficulty of missing information when implementing the asset-equity relation in its original order. The modeling and mathematical details for deriving the CEV based asset-equity relation are provided in Section 2.3 assuming all necessary information is available. The implementation issues of the Monte Carlo based calibration are addressed in Section 2.4. Such an organization allows for temporarily jumping over the last two sections of Chapter 2 if preferred by readers.

2.1 Asset-Equity Relation and the Key Result

The asset value of a limited liability company (LLC) is the value of tangible and intangible resources that can generate positive cash flows. There are different conventions in measuring this value involving accounting or valuation practices. In this study, instead of focusing on any book value of the asset measured by prevailing accounting standards, the market assessment of the asset value is considered. At this stage though, it is not yet fully apparent of how the market can make such an assessment since the assets of a firm are not directly traded. In this section, as a first step, all necessary information, including the market value of asset, is assumed to be available so that the development can focus on analyzing the impact of assets and liability on equity. The discussion of available information and methodology for parameterizing such a model is deferred to Section 2.2.

The market value of assets is modeled as a CEV process, which, under the risk-neutral measure (denoted by \mathbb{Q} -measure hereafter), follows the stochastic differential equation:

$$dV = rVdt + \phi V^\alpha d\tilde{W} \quad (2.1)$$

All the necessary information to parameterize this model, including the initial asset

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

V_0 , CEV volatility ϕ and elasticity α , are assumed to be known at this step. Here r is the risk-free return, which is assumed to be known and constant, and \tilde{W} is the Brownian motion under \mathbb{Q} -measure.

Theoretically, α can take any positive value. When $\alpha = 1$, the CEV model is reduced to GBM, so the CEV model is a generalized version of GBM. It is worth noting that the return distribution skews to the left when $0 < \alpha < 1$ and to the right when $\alpha > 1$. For more details about the CEV model see a summary Hsu et al. (2010).

Cox (1996) introduced CEV model to address the asymmetry in the equity return. Similar but not identical to his celebrated work in 1996, the choice of the CEV model in this thesis as the asset dynamic (instead of GBM) is to introduce asymmetric **asset** return distribution into the structural model. One major limitation of the GBM asset dynamic in leveraged firm modeling is that when leverage is low, the model naturally leads to a flat volatility structure. This is not desirable when considering stocks like Apple Inc. These firms have no essential borrowings compared to their cash holdings, but the stock option market may still suggest a volatility skew (see Section 3.2). Therefore it is not fully valid to credit the entire volatility skew to the leverage effect, and the asymmetric asset return can make an impact as well.

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

An example of calibrating to Apple Inc.'s volatility skew under both GBM and CEV asset dynamic is given in Section 3.2 to justify the benefit of the CEV asset dynamic.

The other major limitation of the GBM asset dynamic is that it cannot produce volatility smiles. Even though implied volatility skews dominate the stock option market, implied volatility smiles can be observed with a non-trivial probability. This once again calls for the introduction of asymmetric asset returns into the structural model (as an alternative to many other improvements to the Black-Scholes model mentioned in Section 1.1). An illustration of calibrating to YAHOO Inc.'s volatility smile is provided in Section 3.1, where one can see the calibration and implications of a smile and thus the value of having the flexibility to accommodate both the skew and smile within one unique model.

Compared to stochastic volatility and jump diffusion models, the CEV model does not introduce an additional source of uncertainty and therefore preserves the market completeness. Compared to the pure jump Levy processes, the CEV model maintains a mathematically tractable form and connects smoothly to prevailing GBM models. This balance between model flexibility and tractability is crucial to simplifying the pricing of vanilla options of equity, which, under the modeling approach in

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

this thesis, becomes a compound option. The compound option is especially difficult to manage because the base layer, i.e. the equity-asset relation, is a perpetual option that cannot be easily approximated by numerical algorithms (an in-depth discussion follows in Section 2.4). To avoid constructing lattices on a perpetual time horizon, a closed-form solution for the equity-asset relation is highly valuable. Such a closed-form solution of $E(V)$ makes it possible to price compound options (i.e. the vanilla options on equity) in a method equivalent to pricing finite maturity barrier options.

Liability modeling is the most important and challenging component of structural modeling. Liability is the major trigger for default, thus it plays a key role in equity valuation. The liability structure can be very complicated in reality, and therefore any mathematically tractable representation cannot accurately replicate real liabilities. Complicated specifications of liability push the models closer to reality but introduce a handful of parameters that cannot be determined with confidence.

Hull et al. (2004a) provided a brand-new perspective on liability modeling. Their idea can be summarized as: modeling liability as a simple structure and mapping the market assessment onto this structure. More concretely, the equity and stock option traders compile detailed information on the liability and cast their opinions into the

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

volatility structure. If a relation between the liability and volatility structure can be established under a manageable liability assumption, this market assessment can then be summarized by this structure. Hull et al. approximated the volatility skew linearly and mapped the linearized volatility skew onto a zero coupon bond maturing at a chosen maturity date. This is accomplished by matching the equity dynamic to the linearized volatility skew.

In this thesis, the nominal liability and its market value are differentiated clearly. The nominal liability, denoted by \tilde{D} , represents the book value of all foreseeable costs for servicing all obligations, discounted to today at the risk-free rate. It is the risk-free value of the firm's liability structure which can be attained only theoretically. In reality, because all firms are subjected to some default risk, the fair market value of liability is always discounted by the default risk. The market value of liability, denoted by D or $D(V)$, is the risk adjusted value of \tilde{D} , which represents the value that market participants are willing to pay when trading the liability structure from one party to another. It is clearly seen later that the market value of liability depends on the asset of the firm.

Differing from Hull et al., a constant perpetual nominal liability structure is

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

adopted in this study. The main reason for choosing this structure is to avoid the specification of maturity, which is not a natural input when examining corporate capital structure. It is assumed to be constant and perpetual because most of the liability structures can be considered as a mixture of two types of basic liability structures which are both (either exactly or approximately) constant in terms of the present value and without a well-defined maturity.

Type one would be an infinite debt with constant coupon, or equivalently infinite annuity which has a constant present value. Once noticing that the return of principle never happens, and annuitizing the perpetual constant coupon by risk-free rate, its present value is strictly constant. This is also the key liability structure considered in Leland (1994). Type two is the rolling zero coupon bond. Even though the present value of such a bond will vary slightly throughout its life cycle due to the risk-free discount, the combination of several bonds with different maturity still yields an approximately constant present value.

Besides these two liability types which motivated the constant liability assumption in this thesis, the preferred stock (also mentioned in Leland (1994)) is also worth noticing. For most of the solvent firms, it is very expensive to alter the structure of

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

preferred stocks thus its constant dividend is more like a perpetual annuity liability rather than a piece of equity. Considering the coupon pressure of the preferred might be a reasonable conservative step to take when evaluating the firm's credit quality. Deeper discussion is seen in Leland (1994).

Similar to the constant interest rate assumption seen in the Black-Schole's model, the constant nominal liability assumption does not fully prevent liability changes, which can be accommodated by updating model parameters. The nominal liability is implied from market data so that the structural model agrees with the stock price and equity volatility. This calibration procedure is discussed in depth in Section 2.2.

The default assumption in this thesis is quite consistent to Leland (1994) due to the similarity in the liability modeling: the default is assumed to be a choice available to equity holders for once, and in a perpetual "American" timeframe. More concretely, the timing of default is assumed to be equity holders' choice in their best interests. When the market value (represented by V) of the firm deteriorates to a certain level, it is no longer the best choice for the equity holders to keep the business running, so they will decide to give away any remaining value of the business to debtholders and walk away with neither benefit claim nor further obligations. This

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

is fully consistent with the limited liability firm organization. Due to the perpetual American nature of the liability, such a default decision can happen for once at any time in the infinite future, and it is hard to mathematically model the timing of human behaviors. However, it is reasonable to assume that equity holders tend to time the default to maximize their own benefit. For a different school of thoughts on default modeling, where the default is triggered by hitting a critical threshold determined by the modeler, see, for example, a paper by Longstaff and Schwartz (1995), in which they provided an analysis of the default risk and interest-rate risk.

Another noticeable feature in the default dynamic assumption is that due to the constant level of total nominal liability and the perpetual nature of the default dynamics, the default decision is made based on the crossing of a constant default barrier L (which depends on the nominal liability level \tilde{D}). This constant barrier greatly simplified the equity option pricing as seen later on. The default dynamics and barrier require careful mathematical treatments, and will be covered in depth in Section 2.3.

Under these assumptions, the asset-equity relation of an LLC under the CEV

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

asset dynamics can be represented by

$$E(V) = V - \tilde{D} + \tilde{D}V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - L^{2-2\alpha})} du \quad (2.2)$$

where the constant default level L can be solved by root searching on the following equation:

$$x - \tilde{D} + \tilde{D}x \int_x^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - x^{2-2\alpha})} du = 0 \quad (2.3)$$

Note that Equation 2.2 is in full compliance to the fundamental relation $V = E + D$. In fact, the derivation of this result is assisted by this relation, and it is shown in Section 2.3 that the market value of liability follows:

$$D(V) = \tilde{D} - \tilde{D}V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - L^{2-2\alpha})} du$$

The last term of equation (2.2) should be the value of a perpetual American put option written on the asset V and with a strike price equal to the nominal liability \tilde{D} under the CEV assumption, because the equity holder is holding not only the net worth $V - \tilde{D}$ but also an additional instrument that allows them to exchange the asset V for an amount equal to nominal liability \tilde{D} , and therefore cancels out the net worth position. This exchange can happen at any time, but is reasonable only when the asset value V falls to the default trigger level L . In such a situation, it is

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

no longer the best interest for equity holders to keep the business in operation, and the redemption of the option allows equity holders to be exempt from any further debt obligations by giving up all remaining business value of the firm. Such an instrument is exactly a perpetual American put option, and its optimal exercise is the endogenous default.

Ekström (2003), the author who first priced the perpetual American put option under CEV assumption, showed that as elasticity α approaches 1,

$$\lim_{\alpha \rightarrow 1} L = \frac{2r}{2r + \phi^2} \tilde{D} \quad (2.4)$$

and

$$\lim_{\alpha \rightarrow 1} \tilde{D} V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - L^{2-2\alpha})} du = (\tilde{D} - L) \left(\frac{V}{L}\right)^{-2r/\phi^2} \quad (2.5)$$

which are the exercise barrier and price of a perpetual American put option under the GBM asset dynamic. This is a highly desirable property because it ensures that $E(V)$ function transits smoothly between the special GMB case ($\alpha = 1$) and the generalized CEV cases ($\alpha \neq 1$). The proof is not reproduced in this thesis.

It is worth comparing the CEV structural model to notable existing ones. First

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

of all, if the perpetual nominal liability assumption is replaced with the Merton's original finite maturity zero-coupon bond liability assumption, the perpetual American put option seen in (2.2) should be replaced by an European put option, and the nominal liability term should be replaced by $e^{-rT}\tilde{D}$. The decomposition seen in (2.2) then becomes $E(V) = V - e^{-rT}\tilde{D} + EuPut(V, \tilde{D}, T)$, which by put-call parity, leads to $E(V) = EuCall(V, \tilde{D}, T)$ that recovers the original Merton's European call model.

If assuming the asset dynamic is GBM, and realizing that a constant perpetual annual coupon C can be annuitized to a perpetual and constant present value, i.e. $\tilde{D} = C/r$, then (2.2) recovers to the no-dividend-no-tax version of Leland (1994) result as seen in (1.14). Therefore, ignoring the dividend and tax components which are not under consideration of this thesis, (2.2) is a generalization to Leland (1994).

2.2 Equity Option Pricing and Market Calibration

Before moving into the discussion of available and missing information for structural model implementation, the equity option pricing method should be illustrated as a necessary building block. A noticeable feature in the structural modeling of

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

corporate leverage is that the equity of a leveraged firm is a derivative, rather than a primitive asset as seen in most mathematical finance modeling. For this reason, the popular equity options (e.g. vanilla options) all become compound options.

Taking a vanilla equity call with strike K as an example, the option price is

$$C_K = \tilde{\mathbb{E}}[B_T(E(V(T)) - K)^+ | \mathcal{F}_t] \quad (2.6)$$

Because C_K is the value of a compound option, the naïve pricing strategy would take two steps. The first step is to find $E(V(T))$ by taking expectation of perpetual option pay-off over all possible future asset paths V (from T to $+\infty$). Then the second step is made to take expectation over all possible $E(V(T))$ to find C_K . This naïve strategy is obviously challenging without the closed-form solution of $E(V(T))$, because of the perpetual American property of $E(V(T))$. To price any perpetual American derivative by Monte Carlo methods, the projection time nodes will need to cover a long enough horizon as well as a fine enough exercise resolution, therefore an intimidatingly large number of projection steps has to be gone through, without speedup techniques (e.g. non-homogeneous projection steps) being readily available. Additionally, it is not enough to process only one or a few sets of perpetual scenarios, because the evaluation of C_K has to take place at a large number of possible

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

$E(V(T))$ to ensure accuracy. Each of this evaluation will require its own simulation to be processed.

Thanks to the closed-form solution (2.2), these challenges are largely reduced, in that the vanilla pricing can be solved by taking only one step of the expectation calculation:

$$\begin{aligned} C_K &= \tilde{\mathbb{E}}[B_T(E(V) - K)^+ | \mathcal{F}_t] \\ &= \tilde{\mathbb{E}}[\mathbb{I}_{\tau > T} B_T(V - (\tilde{D} + K) + \tilde{D}V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^2-2\alpha-L^2-2\alpha)} du)^+ | \mathcal{F}_t] \end{aligned} \quad (2.7)$$

where τ is the first hitting time of V through default barrier L , and $\mathbb{I}_{\tau > T}$ is the indicator function:

$$\mathbb{I}_{\tau > T} = \begin{cases} 1 & \text{when } \tau > T \\ 0 & \text{otherwise} \end{cases} \quad (2.8)$$

The existence of the closed-form solution (2.2) turns a compound option pricing problem into a more approachable down-and-out barrier option pricing, and the remaining difficulty only lies in the complexity of the final pay-off of the barrier option.

Unluckily, a closed-form solution to (2.7) is not immediately available ¹. However,

¹Davydov and Linetsky (2001) provided an excellent study with a closed-form pricing formula for barrier options under CEV specification. However, this work might not contribute to this

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

the situation has been drastically improved in that it can still be solved by Monte Carlo simulation with a reasonable number of steps, i.e. the time horizon is limited and the state variable (V) is clearly identified. Being able to price vanilla options efficiently enables calibration of (2.2) to the financial market, which overcomes the challenge of missing information to be discussed next. However, the runtime issue of this Monte Carlo implementation still has to be addressed. See Section 2.4 for more detailed discussions.

So far the discussion has been based on the assumption that the stochastic process for firm asset value has been fully specified, and all other necessary information is known as well. However, in reality, the related parameters are mostly unknown. As mentioned previously, the assets of the firm are not directly traded on the market, and therefore the initial market value of asset, V_0 , is difficult to observe. Similarly, the asset dynamic parameters, namely CEV volatility ϕ and CEV elasticity α , are even less observable. Also, as implied by Hull et al., structural model implementation is more meaningful if the nominal liability \tilde{D} can be inferred from the market instead of being read from financial statements, in that every accounting standard will intro-

research immediately because the solution derivation depends highly on the simple pay-off structure $(S_T - K)^+$ as seen in, e.g., their equation (21). If repeating this step by integrating the more complex final pay-off needed in this research, i.e. Equation (2.7), the applicability of Davydov and Linetsky (2001) is no longer clear. This research indeed provides a good starting point if further refinement of the vanilla pricing is desired.

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

duce its own bias which will eventually poison the model calibration. Therefore the parameter quadruple $\{V_0, \phi, \tilde{D}, \alpha\}$ is considered as the set of missing information that is better to be inferred from the financial market.

If the only traded asset of a firm is its stock, this calibration will be largely meaningless because it is impossible to determine four parameters from only one constraint. Fortunately in a mature financial market, a handful of equity options are also traded alongside of the stock. The market prices of these options provide rich information about the stochastic dynamic of the firm's equity (e.g. as implied by the Black-Schole's implied volatility smile or skew). Since the structural model attempts to reflect the realistic relation between firm's equity and its fundamentals, equity option prices can serve as a good source of information to infer the unknown parameters.

The actual implementation of the perpetual American CEV leverage model is then a reverse engineering of the relation represented by (2.2) and (2.7) to obtain

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

parameter set $\{V_0, \phi, \tilde{D}, \alpha\}$ that best satisfies

$$\left\{ \begin{array}{l} E_0(V_0, \phi, \tilde{D}, \alpha) = E_0^M \\ C_{K_1}(V_0, \phi, \tilde{D}, \alpha) = C_{K_1}^M \\ \vdots \\ C_{K_n}(V_0, \phi, \tilde{D}, \alpha) = C_{K_n}^M \end{array} \right. \quad (2.9)$$

where the M superscript denotes the market observations of corresponding prices and n denotes the total number of liquid stock options traded on the market. In this way the CEV structural model is calibrated not only to the equity price but also to the implied volatility structure, totaling $n + 1$ constraints.

It is worth noting that, once the model is partially calibrated to the stock price as a first step, the four parameters are no longer fully independent of each other. There are only three degrees of freedom remaining, which are then locked down by the calibration to the set of equity options. That is to say, the volatility skew/smile is calibrated to only three parameters instead of four, largely reducing the concern of over-parameterization. More details are provided in Section 2.4.

2.3 Derivation of the Key Result

As mentioned in the previous chapter, the equity value of an LLC is not simply its net worth $(V - \tilde{D})$ but a more complicated derivative depending on the asset value and the nominal liability. In the next subsection, the CEV leverage model is first formulated into a free-boundary problem, which is then solved in the following section to produce the result seen in (2.2) and (2.3).

Similar to the developments seen in Merton (1974) and Leland (1994), the derivation seen in this section is heuristic rather than rigorous. Some financially related assumptions may post additional challenges when constructing a rigorous proof.

2.3.1 The Free-Boundary Problem Formulation

This subsection will formulate the following free-boundary problem:

$$\left\{ \begin{array}{l} rV \frac{\partial D(V)}{\partial V} + \frac{1}{2} \phi^2 V^{2\alpha} \frac{\partial^2 D(V)}{\partial V^2} + r\tilde{D} = rD(V) \\ D(L) = L \\ D'(L^+) = D'(L^-) = 1 \\ \lim_{V \rightarrow +\infty} D(V) = \tilde{D} \\ E(V) = V - D(V) \end{array} \right. \quad (2.10)$$

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

where L is the optimal default boundary which is an unknown constant before solving the problem. Similar derivation is seen in Leland (1994) under the GBM assumption. It is easier to build the steps by modeling the liability market value $D(V)$. However, it is also feasible to directly model $E(V)$ and the two methods are equivalent. They eventually converge to the same conclusion thanks to the relation $V = E(V) + D(V)$.

The boundary conditions are easier to determine because the behaviors of debt value under extreme situations is almost independent of the stochastic dynamics. The default boundary, denoted by L , is a crucial component in this model. Different treatments to this boundary lead to significantly different models. In this study, we adopt the endogenous default assumption, which means the default boundary is assumed to be selected by the equity holders to maximize their own benefits.

Under the endogenous default assumption, the exact location of the default boundary is not known in advance but needs to be located by additional boundary conditions that must be satisfied at that boundary. It is easy to see that at or below the level of default, the equity holder will retain nothing according to law,

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

thus the debtholder holds the entire firm

$$D(L) = L \tag{2.11}$$

Additional properties of $D'(L)$ can also be discovered. First of all, $D'(L^-) = 1$ because if, by way of contradiction, assume $D'(L^-) \neq 1$, by continuity argument, there exists $\delta_L > 0$ such that $D'(L - \delta_L) \neq 1$. By Mean Value Theorem, there exists Δ_L such that $0 < \delta_L < \Delta_L$ and

$$D'(L - \delta_L) = \frac{D(L) - D(L - \Delta_L)}{\Delta_L} \tag{2.12}$$

Note that for all $V < L$, $D(V) \equiv V$ because of the limited liability law structure.

$$D'(L - \delta_L) = \frac{L - L + \Delta_L}{\Delta_L} = 1 \tag{2.13}$$

This directly contradicts $D'(L - \delta_L) \neq 1$, which is a consequence of the contradiction assumption $D'(L^-) \neq 1$. Therefore the contradictory assumption cannot hold and $D'(L^-) = 1$ has to be true.

On the other hand, $D'(L^+)$ is assumed to be 1 following the derivative assumption

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

of $D(V)$. In order for $D(V)$ to attain its fair value, it has to be dynamically replicable by continuous Delta hedging with the underlying V when $V \geq L$. This requires a continuous $D'(V)$ i.e. a continuous Delta, so that such a replication strategy is still feasible under the basic assumptions of risk-neutral pricing. Therefore, the other key property concerning $D'(V)$ around L can be summarized as

$$D'(L) = D'(L^+) = D'(L^-) = 1 \quad (2.14)$$

When the asset of the firm becomes infinitely large, the market value of liability is assumed to approach its nominal value, because the liability becomes almost risk-less when the leverage becomes of little significance. This is represented by

$$\lim_{V \rightarrow +\infty} D(V) = \tilde{D} \quad (2.15)$$

The Black-Schole's differential equation under the CEV dynamic can be obtained with a method identical to its original (under the GBM assumption) derivation. The key steps are outlined for the purpose of completeness. Starting from the CEV asset dynamic (2.1), the following is easily obtained:

$$dVdV = \phi^2 V^{2\alpha} dt \quad (2.16)$$

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

By Itô's lemma,

$$dD(V) = \frac{\partial D(V)}{\partial t}dt + \frac{\partial D(V)}{\partial V}dV + \frac{1}{2}\frac{\partial^2 D(V)}{\partial V^2}dVdV \quad (2.17)$$

Substituting (2.1), (2.16) and collecting drift and diffusion terms yields

$$dD(V) = \left[\frac{\partial D(V)}{\partial t} + rV\frac{\partial D(V)}{\partial V} + \frac{1}{2}\phi^2V^{2\alpha}\frac{\partial^2 D(V)}{\partial V^2} \right]dt + \phi V^\alpha \frac{\partial D(V)}{\partial V}d\tilde{W} \quad (2.18)$$

To ensure no arbitrage and by the definition of \tilde{D} , under the \mathbb{Q} -measure, the drift rate must follow the risk-free rate less the cost of servicing the liability $r\tilde{D}$, thus yields

$$\frac{\partial D(V)}{\partial t} + rV\frac{\partial D(V)}{\partial V} + \frac{1}{2}\phi^2V^{2\alpha}\frac{\partial^2 D(V)}{\partial V^2} + r\tilde{D} = rD(V) \quad (2.19)$$

Note that due to the constant nominal liability and perpetual American assumption, the system under study is stationary over time, in that there will be no Theta component in the value of $D(V)$. As long as $V_{t_1} = V_{t_2}$ is satisfied, $D_{t_1} = D_{t_2}$ holds regardless of t . This means that $D(V)$ has “no explicit time dependence” (from Leland (1994)), i.e. $\frac{\partial D}{\partial t} = 0$ and therefore (2.19) reduces to its desired form seen in (2.10) (for a similar development on the pricing of perpetual American put options,

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

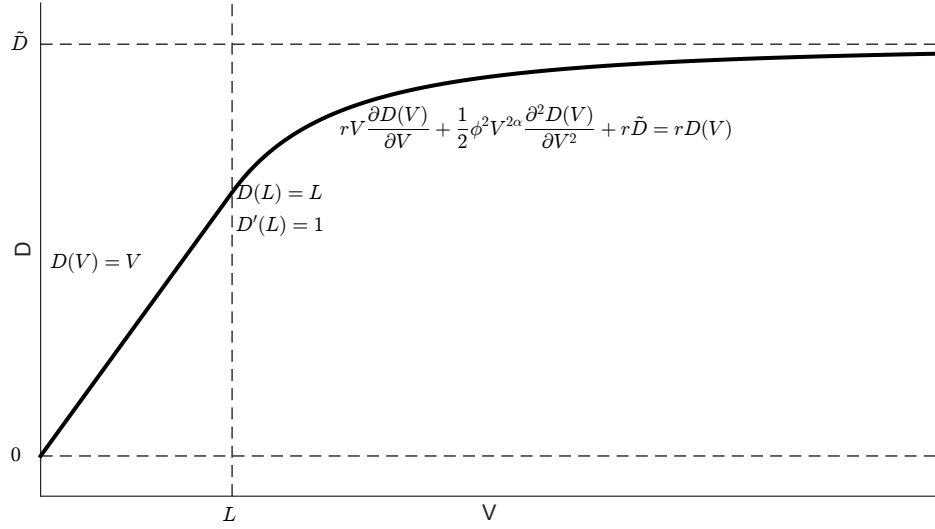


Figure 2.1: Illustration of the free-boundary differential equation and the boundary conditions

see, for example, Section 1.14.2 of Albanese and Campolieti (2006)).

$$rV \frac{\partial D(V)}{\partial V} + \frac{1}{2} \phi^2 V^{2\alpha} \frac{\partial^2 D(V)}{\partial V^2} + r\tilde{D} = rD(V) \quad (2.20)$$

The development of the free-boundary differential equation is now complete.

2.3.2 Solving the Free-Boundary Problem

Equation (2.20) is difficult to work with because it is non-homogeneous. However, through observation, if a solution $M(V)$ solves

$$rV \frac{\partial M(V)}{\partial V} + \frac{1}{2} \phi^2 V^{2\alpha} \frac{\partial^2 M(V)}{\partial V^2} = rM(V) \quad (2.21)$$

then $\tilde{D} + M(V)$ solves (2.20). Therefore the strategy toward solving (2.20) would then be finding the general solution to (2.21), and then using the \tilde{D} shift plus the boundary conditions in (2.10) to determine the specific solution.

Note that since (2.21) is a second order linear equation, its general solution can be represented by linear combinations of two basic solutions:

$$M(V) = \kappa_1 \cdot M_1(V) + \kappa_2 \cdot M_2(V) \quad (2.22)$$

where $M_1(V)$ and $M_2(V)$ denote the two basic solutions and κ_1 and κ_2 are unknown constants to be determined by boundary conditions. Also, if one of the basic solutions is found, the equation can be reduced to first order and therefore becomes much easier to solve.

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

Luckily the first basic solution can be inferred from the financial connotation of (2.21). This equation, which is the stationary Black-Schole's PDE on derivatives of non-dividend-paying underlyings, characterizes the drift-diffusion process that a derivative written on V should follow. The simplest derivative of V is trivially V itself. Therefore V should satisfy (2.21), and this can be verified easily by a substitution exercise. The first basic solution is obtained:

$$M_1(V) = V \quad (2.23)$$

With the first basic solution found, the second basic solution can be found by construction:

$$M_2(V) = M_1(V)g(V) = Vg(V) \quad (2.24)$$

where $g(V)$ is an unknown function that can be solved from

$$\frac{1}{2}\phi^2 V^{1+2\alpha} g'' + (rV^2 + \phi^2 V^{2\alpha})g' = 0 \quad (2.25)$$

From here it is straightforward to obtain:

$$g'(V) = \frac{1}{V^2} e^{-\frac{r}{\phi^2(1-\alpha)}(V^{2-2\alpha} + \kappa_3)} \quad (2.26)$$

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

and

$$g(V) = \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \quad (2.27)$$

Note that it is not worthwhile to discuss the upper limit of integral in (2.27), because for an arbitrary upper limit Ω_u , the following relation always holds:

$$\begin{aligned} \int_V^{\Omega_u} \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du &= \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \\ &\quad - \int_{\Omega_u}^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \end{aligned} \quad (2.28)$$

where the third term is just another constant. Substituting $Vg(V)$ into (2.24) then (2.22), it is easy to see that this constant term makes no contribution because it produces a $\tilde{C}_u V$ term which merges perfectly into the $\kappa_1 V$ term derived from the $\kappa_1 M_1(V)$ term in (2.22) and the first general solution (2.23).

With (2.27) been fully justified, the second general solution is determined:

$$M_2(V) = V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \quad (2.29)$$

and therefore the general form of $D(V)$:

$$D(V) = \tilde{D} + \kappa_1 V + \kappa_2 V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \quad (2.30)$$

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

where κ_1 , κ_2 and κ_3 can be easily determined by the boundary conditions.

First of all, by noting that $\lim_{V \rightarrow +\infty} V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du = 0$, $\lim_{V \rightarrow +\infty} D(V) = \tilde{D} + \kappa_1 V$. Compare to the boundary condition $\lim_{V \rightarrow +\infty} D(V) = \tilde{D}$, $\kappa_1 = 0$ must hold, so that:

$$D(V) = \tilde{D} + \kappa_2 V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du \quad (2.31)$$

The $D(L) = L$ condition leads to:

$$\kappa_2 \int_L^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du = \frac{L - \tilde{D}}{L} \quad (2.32)$$

and $D'(L) = 0$ leads to:

$$\kappa_2 \int_L^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} + \kappa_3)} du - \frac{\kappa_2}{L} e^{-\frac{r}{\phi^2(1-\alpha)}(L^{2-2\alpha} + \kappa_3)} = 1 \quad (2.33)$$

Substituting (2.32) into (2.33) leads to:

$$-\frac{\tilde{D}}{L} = \frac{\kappa_2}{L} e^{-\frac{r}{\phi^2(1-\alpha)}(L^{2-2\alpha} + \kappa_3)} \quad (2.34)$$

which forces $\kappa_2 = -\tilde{D}$ and $\kappa_3 = -L^{2-2\alpha}$. Note that the efforts of proving the uniqueness of κ_2 and κ_3 can be saved, thank to the solution uniqueness of second

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

order linear equation (2.10). These two parameters finally lead to:

$$D(V) = \tilde{D} - \tilde{D}V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - L^{2-2\alpha})} du \quad (2.35)$$

Substituting (2.35) into $E(V) = V - D(V)$ completes the derivation of (2.2) and (2.3).

2.4 Implementation Concerns of the Monte Carlo Based Calibration

Recall in Section 2.2 that the structural model needs to be calibrated to the stock price and a series of stock options (calls as a pick for this research) with a selected maturity. The equity call price (2.7) cannot be solved in closed-form and has to be approximated. This study takes the Monte Carlo approach for approximation, and the search for optimal parameters satisfying (2.9) is carried out by some optimization strategy with nested Monte Carlo approximations.

Before starting the Monte Carlo algorithm, the default barrier L is first calculated by (2.3). This only needs to be done once before it is applied to all scenarios,

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

because the default barrier depends only on the parameters $(D, \phi$ and $\alpha)$ and not on the scenarios. Additionally, the matching of the equity value constraint, i.e. the first equation of (2.9), is also scenario independent and should be carried out before Monte Carlo simulations. In fact, this step helps to reduce the optimization dimension by one, because once $\{\phi, \tilde{D}, \alpha\}$ are given, V_0 is uniquely determined in order to satisfy the first equation of (2.9). Therefore the actual search strategy becomes searching over the $\{\phi, \tilde{D}, \alpha\}$ space by matching the equity value with a proper V_0 and then bringing the option price as close as possible to market prices.

After obtaining the initial asset V_0 , a handful of asset scenario paths will be generated to calculate equity option prices. These paths have to be generated on small time steps because the barrier option nature of the equity requires an accurate identification of the barrier hitting time. For each scenario path, the minimal value of the path is first checked against the default barrier. If the minimum falls below the default barrier, this particular path is marked as defaulted, otherwise the equity value is calculated by (2.2):

$$E(T) = \begin{cases} V - \tilde{D} + \tilde{D}V \int_V^\infty \frac{1}{u^2} e^{-\frac{r}{\phi^2(1-\alpha)}(u^{2-2\alpha} - L^{2-2\alpha})} du & \text{when } \min_t V(t) > L \\ 0 & \text{otherwise} \end{cases} \quad (2.36)$$

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

It is also worth noting that calculating equity value along the scenario paths is unnecessary because only the equity values at a certain future time are relevant to option prices, namely at the maturity of the call options selected as the calibration instruments. After equity values for all paths at option maturity are calculated, the pay-off scenarios for all call options can then be easily determined by $[E(T) - K_i]^+$ for all i , and therefore the vanilla option prices can be calculated. Note that the same set of scenarios is sufficient to price as many vanilla options as necessary.

As described in the previous paragraphs, the majority of computations will happen at scenario generation and at the evaluation of (2.36) across all scenarios. With the development of computing hardware and software, the traditional looping strategy obviously becomes unappealing. Vectorized computing takes advantage of today's ample CPU caches and memories to group several repetitions into one single execution, which can potentially speed up the simulation by multiple times. Indeed, it is the vectorized computing technology makes the implementation of this research feasible at all.

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

The CEV process is discretized by Euler scheme for Monte Carlo simulation:

$$V_{t+1} = V_t + rV_t\Delta t + \phi V_t^\alpha \sqrt{\Delta t}\epsilon \quad (2.37)$$

where ϵ is a standard Gaussian realization. Note that this Euler approximation of the CEV process is naturally vectorize-able and therefore no additional treatment is necessary. It is more challenging to vectorize (2.36) in that the numerical integration is not an easily vectorize-able algorithm. The approach taken in this study is to walk around this difficulty using spline pre-interpolation. Before starting the Monte Carlo simulation, (2.36) is pre-evaluated over a set of V values and later on the evaluation of $E(V)$ becomes a spline interpolation problem, which is easier to vectorize. This strategy is chosen because the $E(V)$ function is relatively smooth over the domain of $[L, +\infty)$ (with smooth components e.g. V and an integral). Furthermore, the interpolation pillar grid does not have to be homogeneous. This is based on the observation that higher-order derivatives (i.e. curvature) concentrate around $V = \tilde{D}$ and the function becomes very close to a straight line when V becomes large enough. Adopting a non-homogeneous interpolation pillar grid will further improve accuracy and reduce runtime.

With all these practical concerns addressed, the algorithm can evaluate one pa-

CHAPTER 2. THEORETICAL DEVELOPMENT OF THE CEV LEVERAGE MODEL

parameter quadruple (namely $\{V_0, \phi, \tilde{D}, \alpha\}$) in a few seconds instead of a few minutes, and therefore carry out calibration for one firm (usually by a few hundred to thousand evaluation iterations) in less than an hour. This paves the way for Chapter 3 where the algorithm is implemented on a larger population of firms so that some interesting results are observed.

Chapter 3

Structural Leverage Model

Empirical Studies

The empirical studies in this chapter features the calibration of model to a volatility smile (rather than a skew), the calibration of volatility skew of a low-leverage firm (AAPL), a rank-correlation study of implied probability of default (PD) and finally a cross-sectional study of implied fundamental characteristics between S&P-100 and NASDAQ-100 component companies.

These four tests illustrate several contributions of the model developed in Chapter 2. First of all, the model is shown to be able to accommodate volatility smiles,

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

which are not compatible to GBM based structural models that can only produce volatility skews. The AAPL fitting example resolves the paradox that GBM models cannot produce profound volatility skew without significant leverage. The implied PD study shows the benefit of implying leverage data from the market versus judgmental processing of financial reports, and the cross-sectional study of the implied fundamental data illustrates the model can produce other metrics which may help understanding firms' financial profiles.

Throughout the chapter, the interest rate is assumed to be constant and flat at 2% for all the tests. All of the market data (equity, volatility cube, CDS curve and financial statements) is collected from Bloomberg. All volatility smiles/skews are sampled at 1.5-year maturity for a balance between time horizon and liquidity. The fitting criterion is the sum of squared relative error of call prices.

The vanilla call prices are simulated by 1,000-path and 1,000-step Monte Carlo, and the Sobol sequence is generated to improve convergence. Quasi Monte Carlo has been a popular strategy to improve simulation convergence and accuracy. However, several researchers have reported loss of performance when the dimension increases (e.g. Moskowitz and Caflisch (1996)). Since the naïve implementation of

1,000-dimension Sobol sequence will be inevitably slow and with limited performance improvements, the Brownian-bridge technique is introduced to reduce the Sobol dimensionality. Paths are generated as a 10-dimension Sobol sequence and then Brownian-bridged to 1,000 steps.

Note that in the CEV model, the volatility parameter ϕ is not a good reflection to the instantaneous volatility. The adjusted volatility $\phi V^{\alpha-1}$ is considered throughout this section, and more discussion is given at the beginning of Section 3.4.

3.1 Reproducing and calibrating to volatility smile

One appealing feature of the CEV-based structural model is its capability of producing different shapes of volatility skew or smiles. It is a valuable feature considering that volatility smile is not compatible to models whose underlying asset dynamic is log-normal.

Figure 3.1 illustrates the volatility skews/smiles under different elasticities (0.9, 1.1, and 1.3), whereas the adjusted volatility ($\phi V^{\alpha-1} = 0.3$) and leverage ($\tilde{D}/V_0 =$

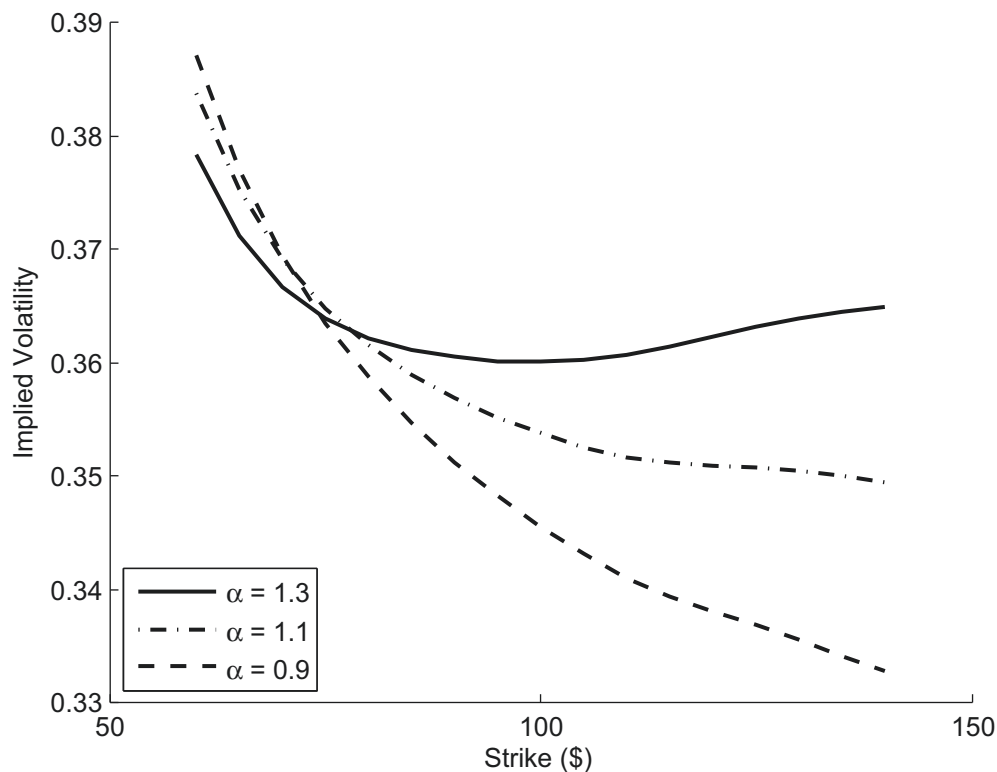


Figure 3.1: Volatility skews and smiles under different elasticity when adjusted volatility is 0.3 and leverage is 0.2

0.2) are held constant. When the elasticity is small, e.g. 0.9, the implied volatilities form a skew, which is commonly observed in the market. As the elasticity increases, the high-strike end of the implied volatility structure tilts up gradually. When elasticity reaches 1.3, the implied volatilities produce an obvious smile, where the implied volatilities are higher for both low and high strikes.

Another noticeable feature is that as the elasticity increases, its impact on the

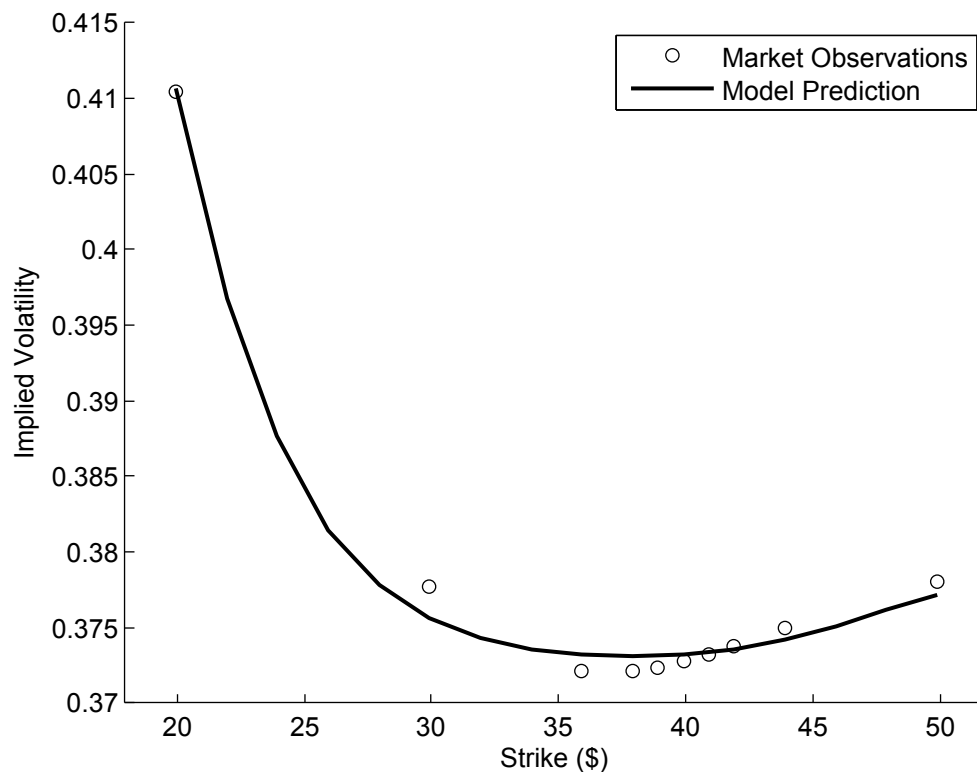


Figure 3.2: Calibration to YAHOO Inc. volatility smile observed on Jan.17, 2014

low-strike end of the implied volatility curve is very limited. This makes intuitive sense, since the low strike options cover scenarios in which the liability and default are the major concern. High strike options, on the other hand, reflect the growth perspective of the firm, which is mostly driven by the firm asset dynamic. It could be loosely interpreted by that the leverage governs the left end of volatility skew/smile, whereas the elasticity governs the right end. This also further validates why both leverage and asymmetry can be extracted from the implied volatilities at the same

time.

Since the 1987 financial crisis, volatility skew has dominated the option market. However, volatility smile can be observed with a non-trivial probability. YAHOO Inc.'s implied volatility on Jan.17 of 2014 is a typical volatility smile example. Figure 3.2 demonstrates how the CEV-based structural model could reproduce this volatility smile with high accuracy. The calibrated model comes with an adjusted volatility 31.36%, implied leverage 16.36% and elasticity 1.3044.

3.2 Volatility skew of low leverage firms

The benefit of picking the CEV process as the asset dynamic is outstanding when modeling low leverage firms. Under the GBM asset dynamic, lower leverage naturally means less volatility skew, which is inconsistent to reality in some circumstances. Apple Inc. (NASDAQ: AAPL) for example, is known for its deep cash position, and therefore its effective leverage is very low. However, a volatility skew is still observed from the option market. The skew is mostly due to the asymmetry of its asset return, rather than to the firm's leverage. If the skew is calibrated to a GBM asset dynamic, the implied leverage must be higher than reality to explain the skew. AAPL skew calibrated under the GBM and the CEV leverage model implies

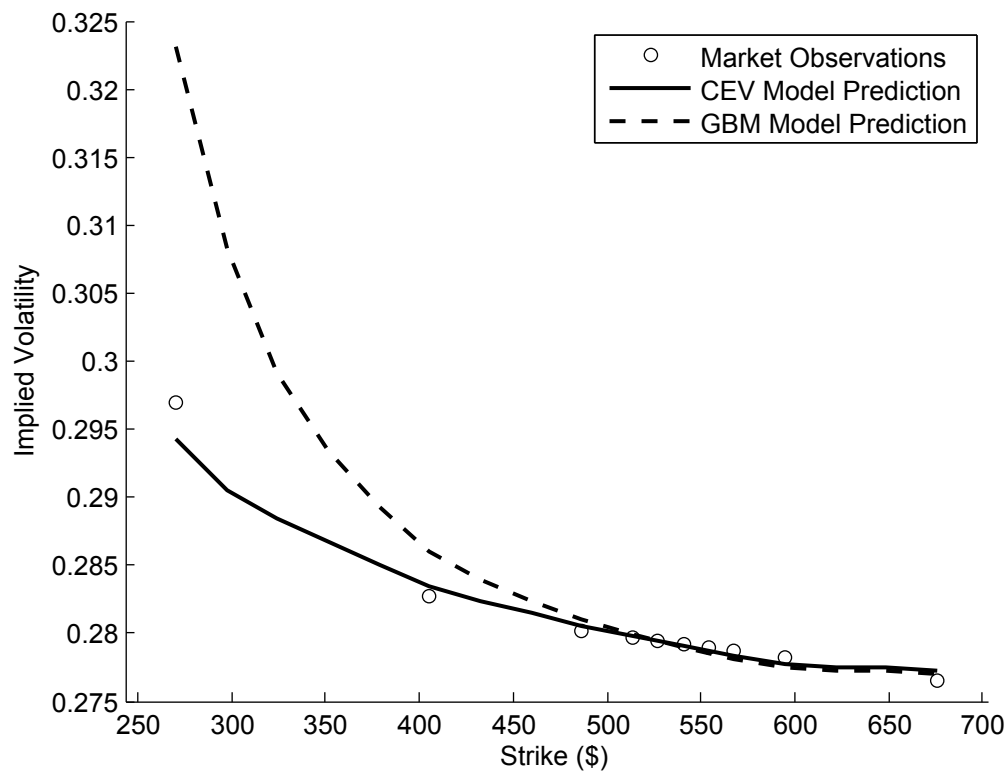


Figure 3.3: AAPL volatility skew calibrated under both CEV and GBM asset dynamic

6.4897% and 1.1166% respectively. From Figure 3.3, it can be easily observed that GBM produces an implied volatility much higher than the market quote on the low strike end. Both the implied leverage and the goodness of fit indicate that the GBM model is overstating the leverage, and the CEV model provides more flexibility to achieve better consistency to the market.

3.3 Rank consistency analysis between expected default loss and CDS spread

The probability of default is another area of interest in the structural modeling, especially when the probability is inferred from the market prices. Such a probability is a highly desirable representation of the market opinion to the firm credit quality. Following the perpetual American assumption of this model, a firm defaults when it hits the default boundary L , and this boundary is absorbing in that once the level is reached, the process stops and remains at that level. This fits exactly into the first hitting time framework. The risk-neutral T -year probability of default (PD_T) of a firm is the probability that the first hitting of default barrier L happens before T :

$$PD_T = \tilde{\mathbb{P}}(\tau_L < T) \tag{3.1}$$

where $\tilde{\mathbb{P}}$ denotes the \mathbb{Q} -measure probability and τ_L denotes the first hitting time to level L .

Under the GBM asset dynamic, the first hitting probability is easier to approach,

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

in that from the GBM asset dynamic

$$dS = \mu S dt + \sigma S dW \quad (3.2)$$

the log-process follows Arithmetic Brownian Motion (ABM):

$$d(\ln S) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW \quad (3.3)$$

The first hitting probability of an ABM has been derived in closed-form (see e.g. Ingersoll (1987)). By realizing that $\ln S$ is a ABM starting at $\ln S_0$, with drift $\mu - \frac{1}{2}\sigma^2$ and diffusion σ , PD_T is then the probability of $\ln S$ hitting $\ln L$ from above by year T which is:

$$\frac{\ln(\frac{S_0}{L})}{\sigma\sqrt{2\pi T^3}} \exp\left(-\frac{(\ln(\frac{S_0}{L}) + (\mu - \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right) \quad (3.4)$$

Under CEV asset dynamic, the first hitting probability calculation is less straightforward, because the CEV counterpart of the ABM does not exist and therefore the trick (3.3) no longer applies. Some semi-closed forms of CEV first hitting probability might exists, but the backward Kolmogorov equation and finite difference method can still serve as a useful and a more general tool to obtain a numerical approximation, without incurring unacceptable runtime.

Kolmogorov backward equation is a powerful tool to calculate the probability of

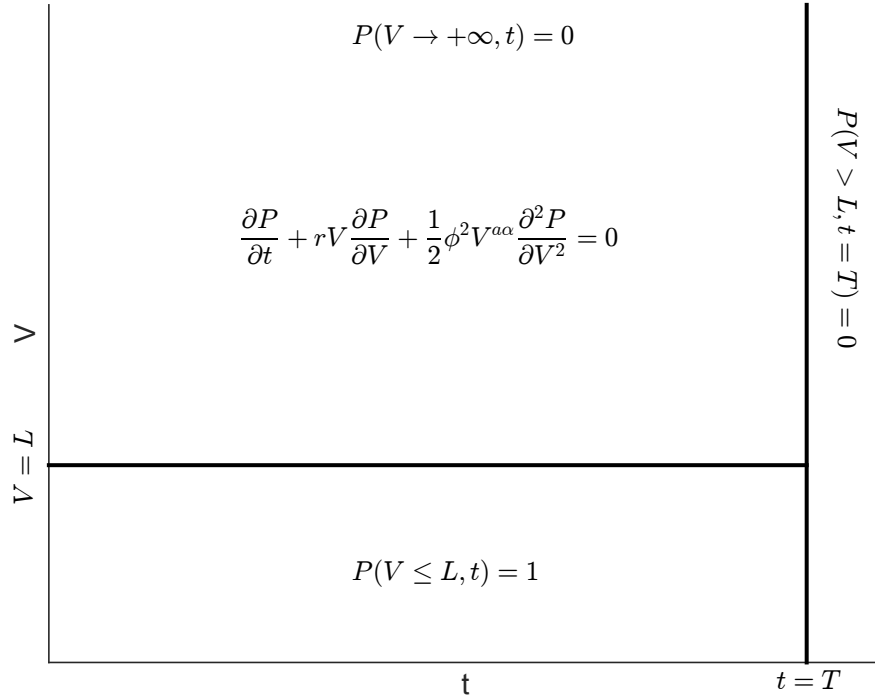


Figure 3.4: Evolution of the probability of default under the framework of Kolmogorov backward equation.

a stochastic process ending with a certain state. The stochastic asset process evolves within the two-dimensional box $0 \leq t \leq T$ and $L \leq V \leq \infty$. The applicability of Kolmogorov backward equation depends on the feasibility of setting up the boundary and terminal conditions in an accurate and manageable format. Because the default barrier L is absorbing, the boundary condition of $P(V = L)$ is one at any time. The box is not bounded from the top and the probability of default when V becomes infinitely large is intuitively zero. The terminal condition is a lump one at the

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

default barrier and zero elsewhere, which joins smoothly with the two boundary conditions. Once the parameters are calibrated, the probability of default (PD) under CEV implementation can be calculated as the first hitting probability by solving the Kolmogorov backward equation:

$$\left\{ \begin{array}{l} \frac{\partial P}{\partial t} + rV \frac{\partial P}{\partial V} + \frac{1}{2} \phi^2 V^{2\alpha} \frac{\partial^2 P}{\partial V^2} = 0 \\ P(V = L, t) = 1 \\ P(V = \infty, t) = 0 \\ P(V, t = T) = \mathbb{I}_{\{V=L\}} \end{array} \right. \quad (3.5)$$

This partial differential equation can be efficiently solved by finite-difference methods, which has absolute advantages in terms of both accuracy and runtime when compared to Monte Carlo simulation for default probabilities. The expected default loss (EDL) is then the product between loss given default $1 - L$ and the probability of default.

It is very challenging to match the implied EDL to CDS spread for several reasons. For one reason, the pricing measure used to generate EDL is calibrated only to vanilla options, and there could be a significant misalignment between the option market and CDS market. A careful joint calibration is required to bridge this

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

gap. Other reasons are also reported in, e.g., Hull et al. (2004b) and Hull et al. (2004a). Therefore an alternative approach of day-by-day analysis is taken instead. A good credit risk measure should be able to produce ranking consistency to CDS spreads, and the movements of CDS spread should also be captured by the credit risk measure. Due to the limited availability of CDS and volatility skew data, it is extremely tedious to apply this test to a large sample. Only two individual stocks with observable CDS movements in the study period are reported here.

To benchmark the effectiveness of the CEV-based structural model, the original Merton's model is implemented under suggestions of Jones et al. (1984) (Merton-JMR) where the asset value (V_0), asset volatility (σ_V) are obtained from equity (E_0), equity volatility (σ_E) and liability (D) input by jointly solving

$$\begin{cases} E_0 = C(V_0, D, \sigma_V) \\ E_0 \sigma_E = \frac{\partial E}{\partial V} V_0 \sigma_V \end{cases} \quad (3.6)$$

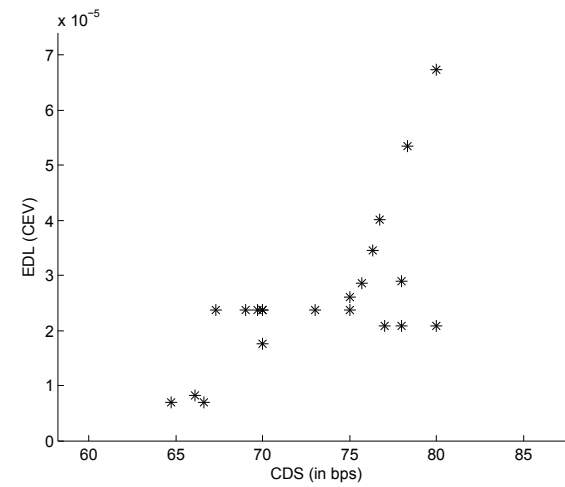
Here the function $C(\cdot)$ represents the Black-Schole's vanilla call valuation, σ_E is approximated by the 30-day realized volatility of the stock return and D is approximated by KMV's suggestion (Crouhy et al. (2000)) of short-term liability plus one half of long-term liability.

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

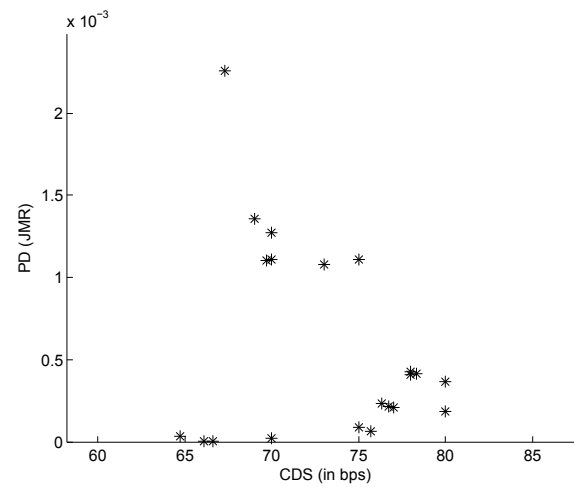
The market data of American International Group Inc. (NYSE:AIG) and Simon Property Group Inc. (NYSE:SPG) between Jan.17 of 2014 and Feb.28 of 2014 are analyzed. Figure 3.5a and 3.6a show the scatter plots of 5-year EDL v.s. 5-year CDS spread under CEV implementation for AIG and SPG respectively, whereas Figure 3.5b and 3.6b show the scatter plots of 5-year probability of default v.s. 5-year CDS spread under Merton's JMR implementation for AIG and SPG respectively. It is quite noticeable that under CEV implementation, the EDL shows a much stronger ranking consistency with CDS, compared to the PD under JMR implementation.

Two different rank correlation measures are commonly cited, namely Kendall's and Spearman's. Because the sample size is relatively small and tends to contain noise and errors, the Kendall rank correlation is selected because of its robustness toward small and noisy samples. The Kendall rank correlation measure confirms the visual observations: CEV implementation leads with 0.5327 Kendall τ v.s. JMR's -0.2284 in the SPG case, and 0.5045 v.s. -0.0290 in the AIG case. This suggests that for a particular firm, the EDL produced by CEV implementation works more desirably in capturing the daily movements of the CDS spreads.

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES



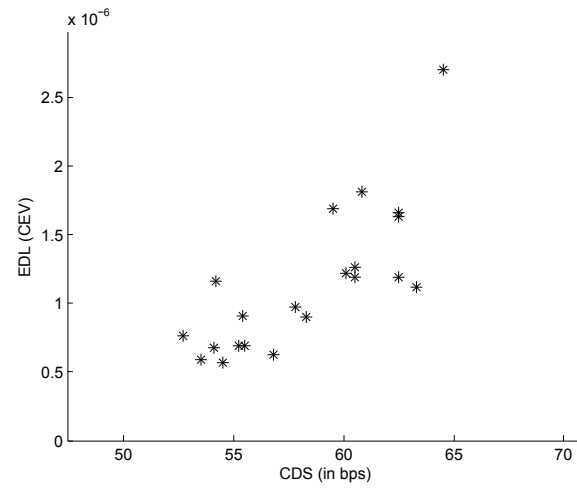
(a)



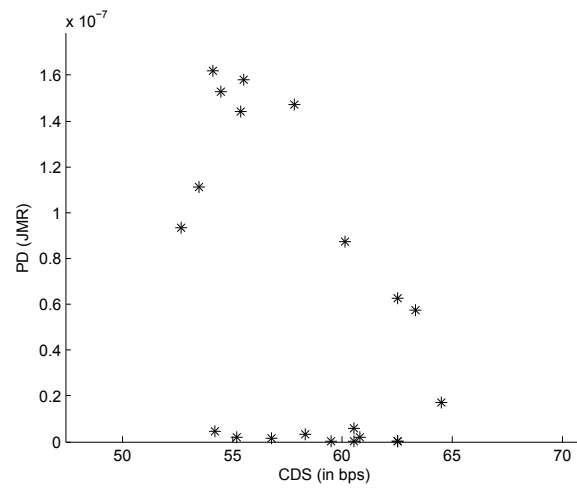
(b)

Figure 3.5: Relationship between (a) CEV EDL (b) JMR PD and CDS, NYSE:AIG

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES



(a)



(b)

Figure 3.6: Relationship between (a) CEV EDL (b) JMR PD and CDS, NYSE:SPG

3.4 Additional Challenges in Cross-sectional Study of CEV-based Structural Model

All of the developments in Chapter 2 hold true when applied to one single firm. However, it is natural to extend the CEV-based structural model to a group of firms and carry out cross-sectional analysis. This creates no additional challenge when the firm dynamic is modeled as a Geometric Brown motion, because the GBM instantaneous asset return follows:

$$\frac{dV}{V} = rdt + \sigma d\tilde{W} \quad (3.7)$$

Once the only parameter, σ , is fitted, it is fair to compare between different firms because σ itself fully specifies the volatility of the firm asset return.

Similar arguments no longer hold when the asset dynamic modeling moves from GBM to CEV. Under CEV asset assumption, the instantaneous asset return dynamic becomes:

$$\frac{dV}{V} = rdt + \phi V^{\alpha-1} d\tilde{W} \quad (3.8)$$

This equation immediately reveals the problem. The instantaneous volatility, $\phi V^{\alpha-1}$,

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

is not fully specified by the CEV volatility ϕ . Even worse, the volatility cannot be fully specified by a combination of several parameters because it also depends on the level of asset. This makes it even harder to compare the asset return asymmetry because the contribution of the elasticity parameter α is coupled by the level of V . This is a major disappointment for CEV model, in that if the same asset stochastic process is calibrated under different units of denomination values (e.g. cents, dollars, million of dollars, etc), different elasticity α will be produced even though they all represent the same level of return asymmetry.

The cross-sectional asset volatility comparison can be easily achieved by comparing the adjusted volatility $\phi V_0^{\alpha-1}$. This adjustment also makes the volatilities from the CEV model comparable to GBM volatilities. Such a remedy for the elasticity analysis is not readily available. To overcome this difficulty, the cross-sectional comparison of α is based on a normalized asset value. For each firm, the total equity is re-calculated based on the assumption of \$1 million asset, and all of the traded options are adjusted so that the adjusted option strikes are still aligned to the adjusted equity price in terms of strike-spot ratio:

$$\frac{K_{\text{Adjusted}}}{E_{\text{Adjusted}}} = \frac{K_{\text{Actual}}}{E_{\text{Actual}}} \quad (3.9)$$

In this way, the market implied volatilities are still meaningful. All candidates in the population have the same asset value and therefore the comparison of α becomes more valid.

3.5 Cross-sectional Parameter Distribution Analysis

The ideal test of effectiveness of implied parameters (i.e. V_0 , \tilde{D} , ϕ and α) is comparing the parameters to fundamental research conclusions. However this analysis is difficult to approach because the information and workload required for fundamental research is far beyond the author's resources and specialties. On the other hand, the conclusions of fundamental research are difficult to quantify as well.

To overcome this difficulty, the cross-sectional approach as described in Section 3.4 is taken as an alternative. 99 components of S&P-100 index and 81 components of NASDAQ- 100 index (with 17 overlaps excluded) are selected as test candidates to ensure strong capitalization, market liquidity and data accuracy. Candidates are dropped either because of missing market data or because of index overlap. From the fundamental point of view, components of two indices should be notably differ-

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

ent because S&P-100 components are mostly mature and stable companies, whereas the NASDAQ-100 components have a much higher emphasis on high risk and high growth potential stocks. These two types of companies should differ in terms of business uncertainty, effective leverage and asset return asymmetry.

Similar to Section 3.3, the CEV implementation is benchmarked by the Merton-JMR implementation. The distribution comparison is based on kernel smoothing function estimation. This choice is based on the adaptability of kernel smoothing function. Since kernel smoothing is not based on any assumption of the distribution and is fully non-parametric, it is extremely suitable to this study because the distribution of parameters within a population is never studied. On the other hand, all three parameters, namely volatility, leverage and asset return asymmetry, are reasonably bounded and thus have natural supports to adopt. All three parameters are fitted with normal kernels, and both asset volatility and leverage are constructed on the support of $[0, 1]$. The return asymmetry kernel estimation is built on support of $[0.4, 1.5]$.

The difference between distributions is measured by the Hellinger distance to overcome possible outlier problems commonly seen in financial data. The Hellinger

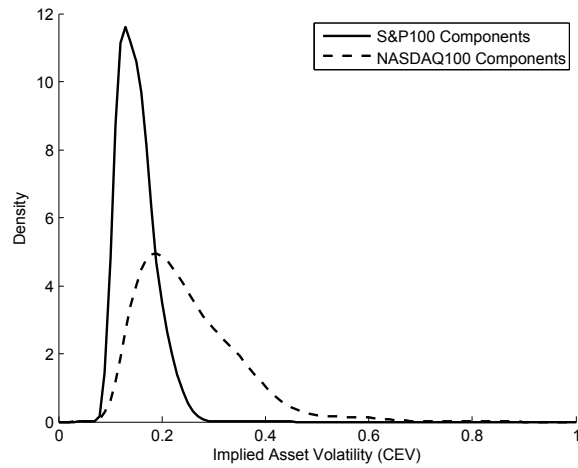
CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

distance between density function $f(x)$ and $g(x)$ is defined as:

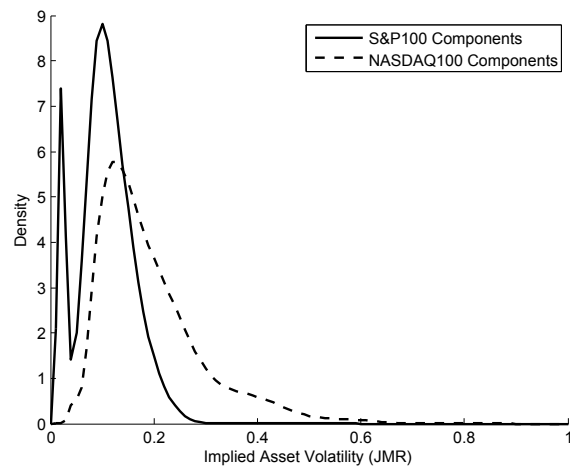
$$H < f, g > = 1 - \int \sqrt{f(x)g(x)}dx$$

Unlike in GBM, the CEV volatility factor ϕ is not directly comparable between candidates because of different α and V_0 . The adjusted volatility $\sigma = \phi V_0^{\alpha-1}$, which is the instantaneous volatility of asset return, is compared instead. Figure 3.7 shows the Kernel smoothing function estimations of asset volatility under both CEV and JMR implementations. It is remarkable that the adjusted asset volatility of S&P-100 companies clusters around 10%, whereas for NASDAQ-100 companies it is more dispersed to the higher volatility zone and with a fat tail on the right. This is consistent with the fact that stable businesses usually have lower and similar uncertainty, whereas growing businesses tend to have higher and more dispersed volatility due to the business model diversity. The Merton-JMR implementation captures a similar pattern, but with much lower confidence. The Hellinger distribution distance between S&P-100 and NASDAQ-100 for the CEV implementation is 0.2899 compared to 0.2022 for the JMR implementation. Therefore, both implementations correctly capture the differences in business uncertainty, while CEV implementation demonstrates an advantage in producing more differentiated volatility distributions.

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES



(a)



(b)

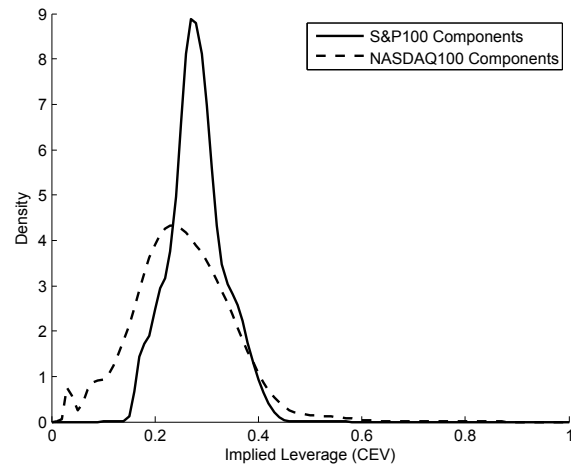
Figure 3.7: Kernel smoothing function estimation of firm volatility under (a) CEV (b) JMR implementation

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

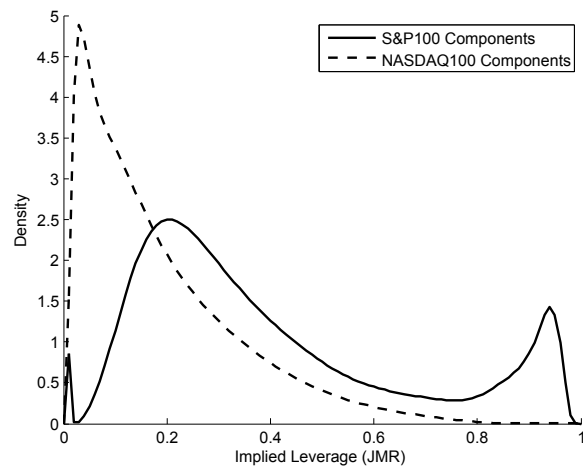
One major difference between CEV and Merton-JMR implementation is the treatment of liabilities. Liabilities are readable from financial reports. However these values are highly unreliable in many respects. For example, stable companies tend to make long-term rolling borrowings, which pump up their liability book values without putting them under significant financial stress. These liabilities are usually adjusted down in fundamental analysis. Growing companies, on the other hand, tend to make short-term borrowings, and their capability to roll over these debts depends highly on their short-term performance. These liabilities are usually the default triggers for growing companies and should not be adjusted down. Off-balance-sheet items further complicate the liability analysis by its hidden and diversified nature.

The CEV implementation in this thesis is dedicated to implying nominal liability from market data, and therefore takes into account the professional adjustments made by fundamental traders. JMR implementation can only make very crude estimates. Figure 3.8 shows the Kernel smoothing function estimation comparison of implied leverage under CEV and JMR implementation respectively. Once again, CEV implementation suggests concentrated leverage distribution for S&P-100 companies and more dispersed distribution for NASDAQ-100 companies. Additionally, NASDAQ-100 companies show slightly lower leverage, which is also consistent with

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES



(a)



(b)

Figure 3.8: Kernel smoothing function estimation of implied firm leverage under (a) CEV (b) JMR implementation

CHAPTER 3. STRUCTURAL LEVERAGE MODEL EMPIRICAL STUDIES

reality. The JMR implementation produces very different results in terms of leverage. A significant number of S&P-100 companies have leverage close to one, and NASDAQ-100 companies show high concentration at very low leverage. This might be true when reading the financial statements, but the implication is less meaningful or even misleading when trying to understand the firm's fundamentals. Even though JMR implementation shows a higher Hellinger distribution distance than CEV (0.1904 vs 0.1193), it should not be considered an advantage due to its misleading implication.

The distribution of elasticity factor α is also illustrative in revealing the firm characteristic. Figure 3.9 shows the Kernel smoothing function estimation under CEV implementation (note that this measure is unavailable under JMR implementation). For NASDAQ-100 stocks, a strong clustering around $\alpha = 0.7$ and a weaker clustering around $\alpha = 1.2$ are very noticeable. The strong clustering to lower elasticity illustrates that a fat left tail in asset return distribution is expected by the market for most of the growth company, whose valuation is rich and downside is large; whereas the weaker clustering to higher elasticity corresponds to the fewer companies with moderate current valuation but with strong growth potential which has not been richly priced. The distribution of S&P-100 asset return elasticity is more dispersed,

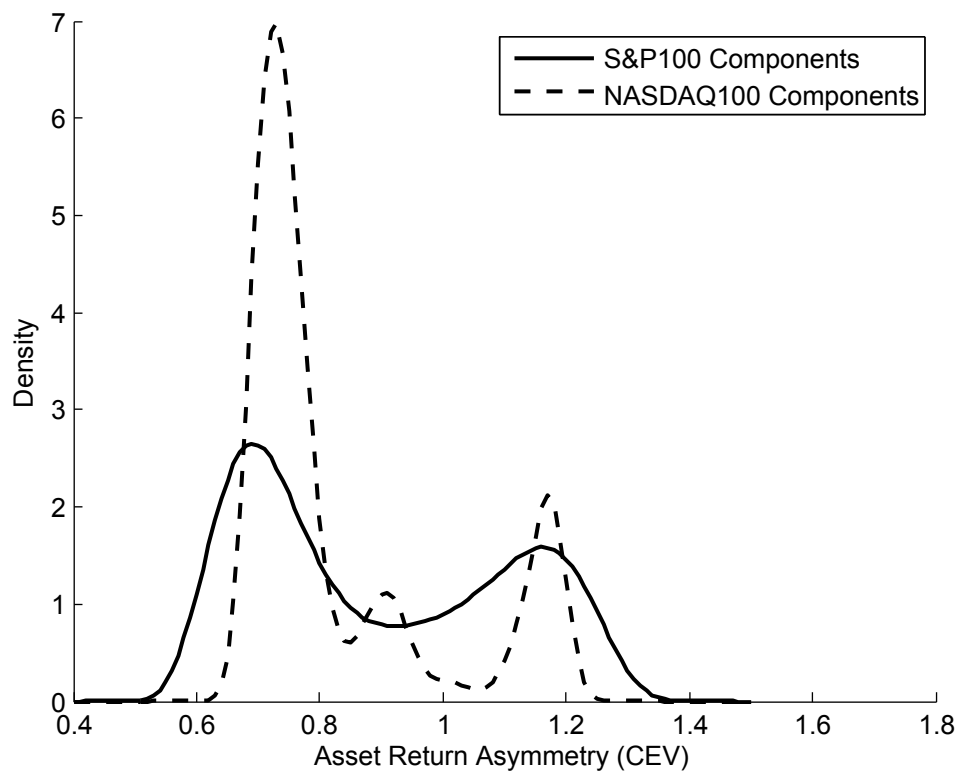


Figure 3.9: Kernel smoothing function estimation of asset return elasticity (α) for CEV implementation

which is consistent to the fact that these stocks are mostly fairly valued, and the elasticity purely reflects the nature of the business' profitability. The Hellinger distance between elasticities is 0.1407.

Chapter 4

An Investment Protection Overlay based on Perpetual American Put Option

4.1 Protection Overlay Introduction

Fund protection strategies were initially popular in the fund of hedge funds management. Recently, a growing popularity is seen in retail investment businesses including mutual funds, pension and especially variable annuities (VA). Various strategies were introduced but the outcomes are very similar: fund protection overlays

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

change the base assets' return profiles, and therefore produce a new asset which cannot be easily replicated by the base assets. This has a profound impact on market players who handle derivative contracts written on these protected assets. VA hedgers, for example, write complex put options on the funds that retail customers are choosing. With the introduction of protection overlay, VA hedgers are immediately in an uncomfortable position of writing options on less liquid underlying assets whose behaviors are not fully characterized.

This chapter presents a fully systematic fund overlay strategy based on the perpetual American put (PAP) option. A static hedging strategy is provided in accompany as an alternative perspective to hedging derivatives written on protected funds. Such a static hedging strategy can also serve as a measurement tool to evaluate the effectiveness of various protection overlays on the market.

Compared to some existing counterparts, the PAP overlay is shown to provide a strong protection against the extreme loss events in terms of both ending account values and deepest drops. Deepest drop protection is as important as ending value protection because almost all VA products are embedded with some early exercise privileges, so that a deep drop before maturity (or annuitization) can be as harmful

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

as a big loss in the ending account value. At the same time, the PAP overlay is much less path-dependent, providing a significant cost relief to scenario based risk-management practices widely seen in VA hedging. Being a systematic strategy is also a key contribution to fund management, because behaviors of protected funds are fully predictable once the base asset returns are prescribed. This is highly preferred to VA hedgers in that they have a better understanding to the assets they are selling options on, therefore taking a key step towards a better management of the residual risk in the protected assets.

This being said, hedging derivatives written on protected assets remains challenging because the protected assets do not have a deep enough market to accommodate long/short positions. There is virtually no way to carry out traditional dynamic re-balancing, which is only practical when the primitive asset is highly liquid and tradeable. Therefore the proposed static hedging strategy becomes an interesting alternative to battle this challenge by making the risk management and performance assessment more feasible.

4.2 Existing strategies

Most of the existing protection overlay strategies are proprietarily owned by fund managers. However a majority of them can be roughly illustrated by three simple prototypes, i.e. volatility targeting, option purchase and option replication.

Volatility targeting predicts the realized volatility of the base asset in the next rebalancing interval, and therefore allocates between risky asset and cash accordingly to achieve the targeted volatility (σ_t). One simple example is using 30-day rolling realized volatility (σ_m) as the volatility forecast and setting the equity allocation as σ_t/σ_m . More sophisticated algorithms (e.g. GARCH or EWMA) can be implemented to improve the volatility forecasting. However, this algorithm will suffer when the market is in a progressive market crash. The volatility forecast stays low while the stock price falls, so the control of volatility level cannot address the loss of investment value. One extreme example is demonstrated in Figure 4.1.

Fund managers can also allocate between the base asset and its corresponding options. The simplest pick of the option would be the 3-month at the money put ($P(0.25, 1)$), which is rolled into a new put at each maturity. Assuming that the per-unit value of the base asset is S and the corresponding put price is P , the fund is then allocated as $\frac{S}{S+P}$ into the base asset and $\frac{P}{S+P}$ into the options at each rebal-

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

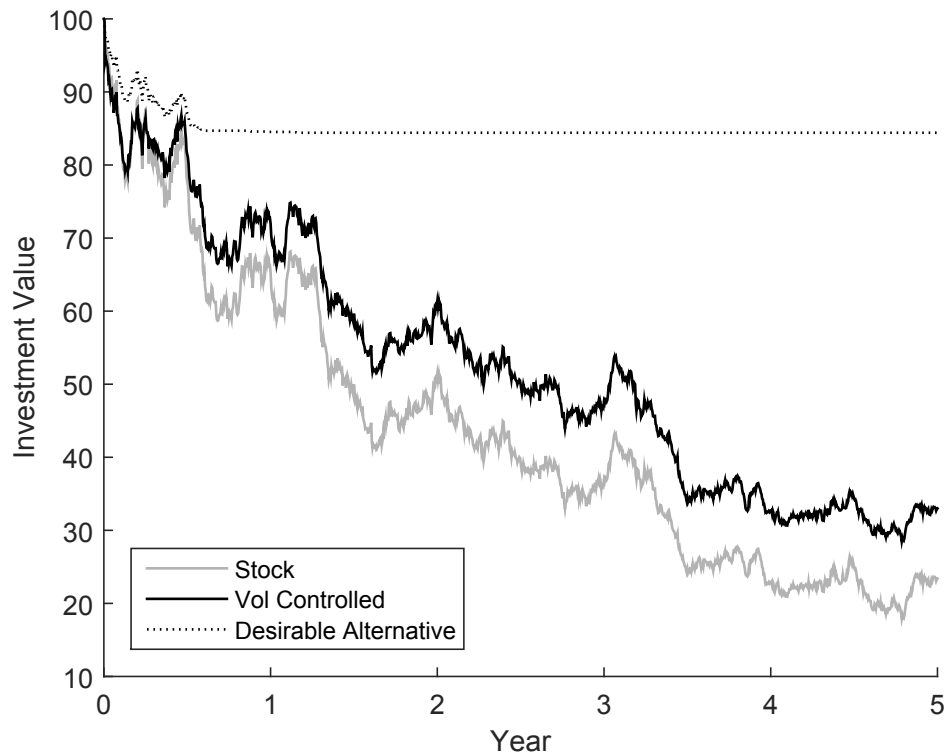


Figure 4.1: The performance of volatility controlled fund under progressive market crash, with a more desirable alternative.

ancing. This can be an expensive strategy due to the fact that put option sellers will usually charge a premium as their business profits, and transaction costs of options are usually high as well. Another potential weakness is that the choice of option strikes and expirations is usually very limited due to the poor liquidity of option contracts with long maturities or heavily in-the-money/out-of-the-money strikes.

The option purchase strategy can be improved by replacing the market options

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

with dynamic replication based on option Delta, which evolves into a third strategy. The fund manager will enjoy more flexibility in option specifications (e.g. strike and maturity) since the option liquidity is no longer a concern. Typically the desired options provide protections to the downside of the base asset, thus usually possessing a negative Delta. The negative Delta neutralizes the allocation to the base asset by allocating to cash and dynamically readjusts this allocation according to market conditions. With the previous put purchase example, assuming the Delta of the option is $-\Delta$ (where Δ is a positive number), then the option can be replicated by shorting ΔS dollar of the base asset and allocating $P + \Delta S$ dollar into cash. The overall dollar allocation between stock and cash should then be $S - \Delta S$ versus $P + \Delta S$. The key challenge in this strategy is to obtain a robust estimation of Delta. One typical practice is marking to the market: using the market implied volatility surface to interpolate for the best implied volatility and then calling the Black-Scholes formulas. In this way the strategy is subjected to significant model and parameter risk, especially Vega and Rho risk, in that the yield curve and implied volatility surface swing from day to day, introducing significant instability to the strategy and therefore execution losses.

4.3 PAP-based protection overlay

The perpetual American put model for leveraged firms serves as a good inspiration to the overlay strategy design, and also falls nicely into the third category of protection strategy, namely option replication. The two non-constant terms, $V + Put(V)$ provide a desirable remapping of V 's return profile, as illustrated in Figure 4.2. When the base asset's value is high above the strike, the protected asset's return becomes very close to the base asset, so that the investment participation on the upside is very high. On the other hand, as the base asset's value starts to decline, the protected asset starts to pull out of the base asset (technical detail follows) thus reducing the participation and therefore creating downside protection. An impressive feature is that when the base asset falls to a certain threshold, the strategy will pull completely out of the base asset and remain fully protected until the base asset recovers to above this threshold. This suggests that allocating between the base asset and a dynamically replicated perpetual American put option can work as an appealing protection strategy.

Given a base asset S , the overlay can be implemented alongside the holding of S by dynamically replicating a perpetual American put option with underlying as S and G defined by the risk appetite of the fund manager at rebalancing (e.g. the

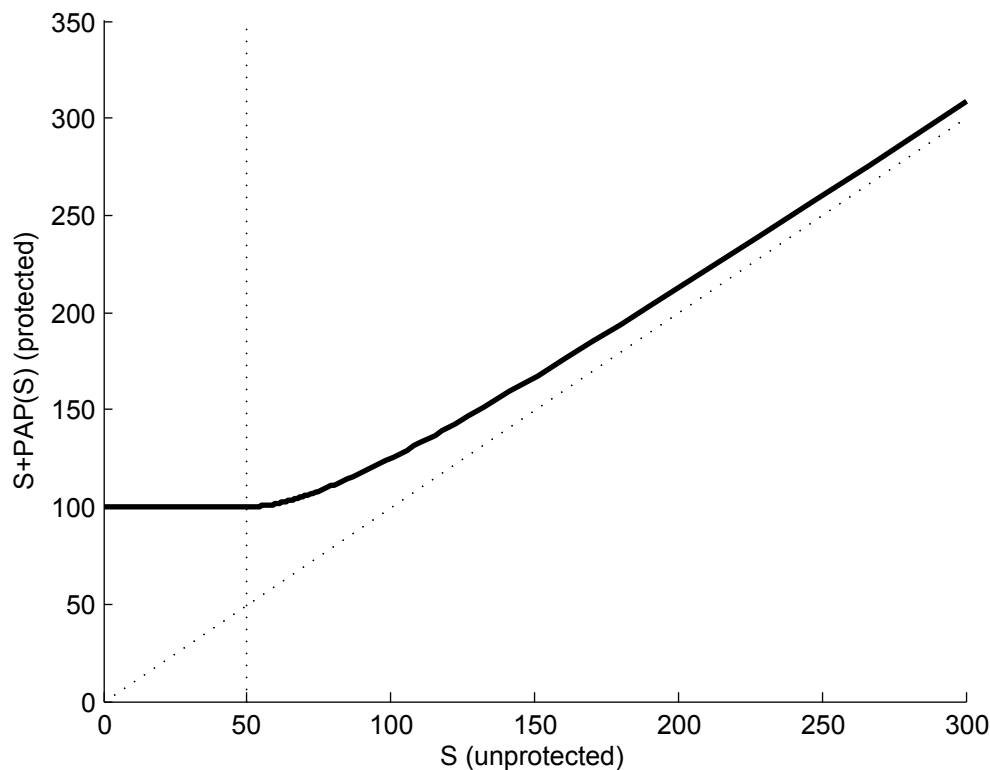


Figure 4.2: An example of the price relation between an unprotected fund and a PAP-protected fund: \$100 strike, 2% interest rate and 20% volatility

initial asset level S_0 or the rebalancing asset level S_t). Given the perpetual nature of the option, the risk-free rate and asset volatility can be chosen with higher confidence as the long-term mean-reversion level, eliminating the parameterization risk concerns.

Under the GBM assumption on S , the mandatory pull out level:

$$L = \frac{2r}{2r + \sigma^2} G \quad (4.1)$$

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

is the threshold where fund manager should be fully de-allocated from the risky asset.

The value of the perpetual American put option is:

$$P = \begin{cases} G - S & (0 \leq S \leq L) \\ (G - L) \left(\frac{S}{L}\right)^{-\frac{2r}{\sigma^2}} & (S \geq L) \end{cases} \quad (4.2)$$

and the option Delta is:

$$\Delta = \begin{cases} -1 & (0 \leq S \leq L) \\ -\frac{2r}{\sigma^2 S} (G - L) \left(\frac{S}{L}\right)^{-\frac{2r}{\sigma^2}} & (S \geq L) \end{cases} \quad (4.3)$$

Note that Δ is continuous around threshold L , which is a very desirable property to make the strategy practical, meaning that the Delta-based trading can be continuously executed when the base asset value falls below L from above. The option can be dynamically replicated by shorting $-\Delta$ shares of the base asset and allocating $P + \Delta \cdot S$ into cash. The short position of base asset neutralizes the allocation to the based asset S , leaving the portfolio allocated in proportion to:

Asset	Cash
$S + \Delta \cdot S$	$P - \Delta \cdot S$

A simple version of this strategy is implemented to demonstrate its protection

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

effectiveness. The base asset is the SPDR S&P 500 ETF (NYSEARCA: SPY). The strike of the perpetual American put option is always set to equal the initial investment, which is assumed to be \$100. Both the volatility and the risk-free rate are assumed to be approximately their long-term mean reversion level, i.e. 0.2 and 0.04 respectively. The rebalancing happens at the end of every trading date, and the 2002 and 2008 data is tested.

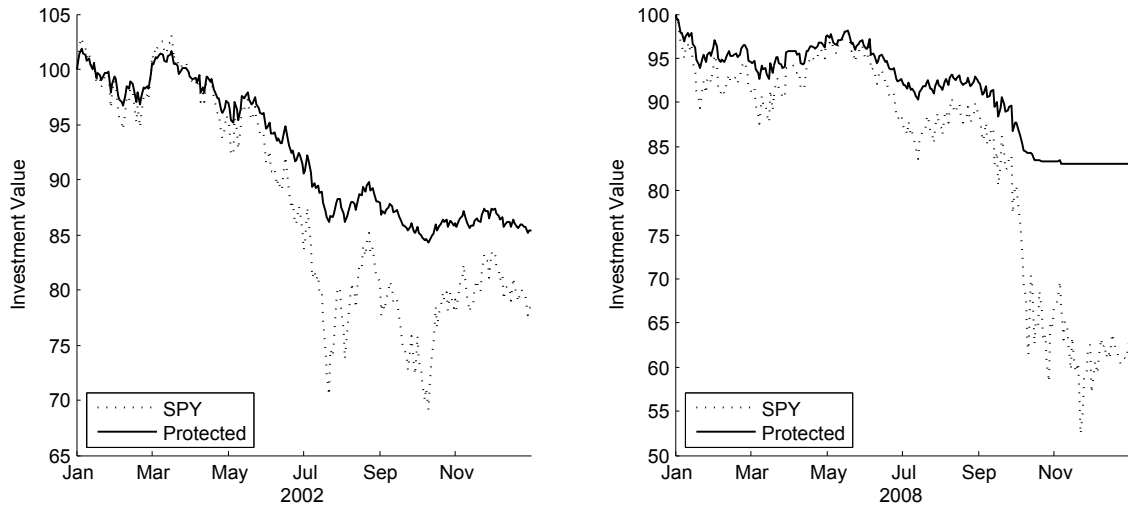


Figure 4.3: Historical performance of perpetual American put overlay in 2002 and 2008

Table 4.1: PAP protection's impact on annual returns and maximum losses

Year	Annual Return		Max. Loss	
	SPY	Protected	SPY	Protected
2002	-24.32%	-13.57%	-37.00%	-15.43%
2008	-45.85%	-16.12%	-64.58%	-16.70%

4.4 Impact of protection overlay on hedging

The two particular historical observations demonstrate the effectiveness of put replication protection under crude parameter specifications. The mandatory pull-out boundary of perpetual American put option sets up a strong control on the falling asset level. This kind of protection should, in return, assist hedging rather than purely complicating it. With an effective protection in place, hedgers who sell options on the protected asset do not need to maintain a fully-loaded and dynamic hedging position because the protected asset has less downside risk than the base asset. If an identical put option is written on both the base asset and the protected asset, the hedger should be able to hedge the option written on the protected asset with a fractional cost rather than hedging options written on the base asset, given that the protection strategy is a successful one.

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

Motivated by the previous reasoning, a hypothetical static hedging approach is presented in this section. The static strategy allows the hedger to enter any desired vanilla put option position at the beginning, with the strike and percent of coverage (fraction of vanilla per share) of put at the hedger's choices. Taking a simple but illustrative example of hedging an at-the-money put option on the protected asset, if the base asset is not protected at all, a full strike and a full coverage put must be selected to achieve the hedging target (in this example a zero expected payout shortfall), which corresponds to point A in the Figure 4.4. If the protection performs perfectly to eliminate all of the downside, there is no desire of purchasing any option, i.e. both the strike and the coverage should be zero. This corresponds to point B in the figure. A realistic protection strategy will call for put purchases with combinations of strike and percent of coverage between A and B to achieve the hedging target, while better strategy will push the combinations closer to the southwest of the box. One protection strategy can be hedged by different combinations between strikes and percent of coverage, forming a curve named as the hedge cost frontier which is discussed later.

To further evaluate how the hedging target has been met, it is insufficient to test the overlay on a few historical scenarios. The more desirable approach is to cycle

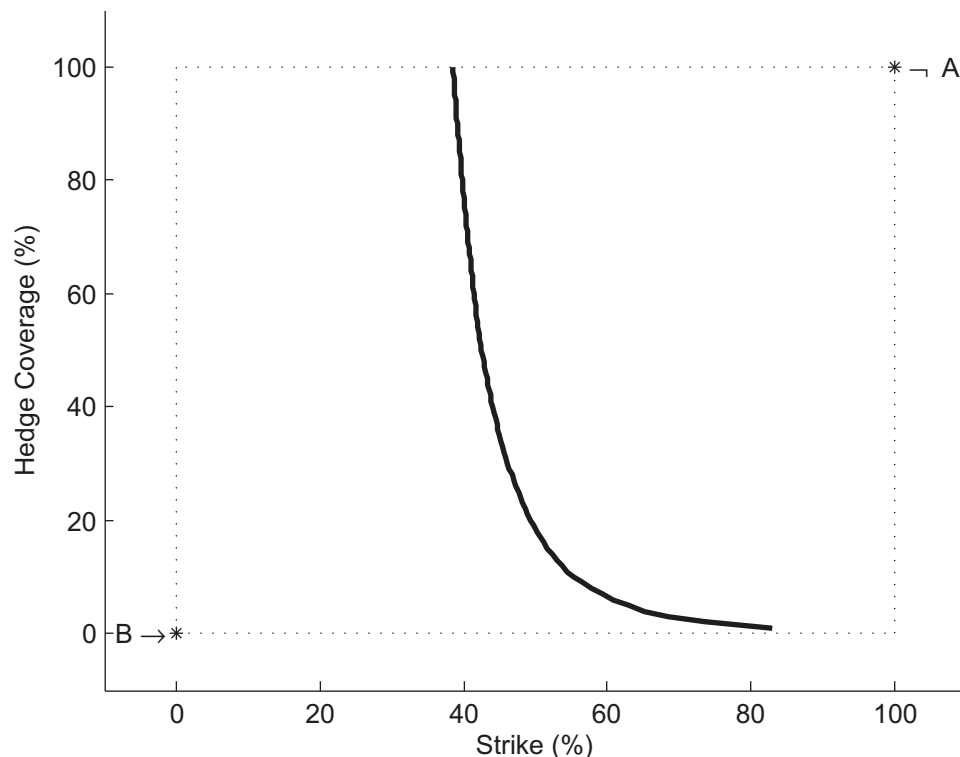


Figure 4.4: One example of the hedge cost frontier. Each point on the frontier indicates that to achieve the desired hedging goal, a put purchase with the combination of fractional strike ($\frac{\text{Put Strike}}{\text{Initial Value}}$) and fractional shares ($\frac{\text{Share of Puts Purchased}}{\text{Shares of Investments}}$) is required.

through a much larger sample of paths. One possible misconception could be evaluating the protection effectiveness over the marked-to-market risk-neutral scenarios. Risk-neutral simulation is not relevant in this application because the purpose is not to achieve any market price consistency by calculating expectation under a proper probability measure. The market consistency and risk neutrality will not harm the simulation, but might introduce bias since it is not a good representation of what might occur in the course of hedging. To better stress test the protection effec-

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

tiveness, a real-world simulation approach is more pertinent. Since the real-world scenario generation by itself can lead to a lengthy discussion which is not the major focus of this study, a simpler approach is adopted, namely the historical maximum likelihood estimation of a Heston stochastic volatility model. It is helpful to recognize that the flexibility in the choice of simulation scenarios depends fully on the manager's interest. If the manager is more concerned about the highly stressful scenarios, then one could fit historical model to stressful years, e.g. 2008. It is also fully valid to mix scenarios of different flavors which reflect the overall risk management considerations.

The methodology for historical maximum likelihood estimation of Heston model is provided in Appendix C. The S&P total return daily index from 2005 to 2010 was used to fit the Heston model, and 10,000 five-year daily paths were generated as the forward-looking scenarios. On each scenario path, the perpetual American put based strategy was executed with 70% strike (\$70), 20% volatility assumption and 4% interest rate assumption. At each trading day, the account grows according to the stock return and its stock allocation. Because the 4% interest rate assumption is relatively aggressive which cannot be easily achieved on a daily basis, a highly conservative adjustment is made so that the cash allocation does not grow throughout the

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

entire process. This will dampen the strategy performance by some extent, but the overall result will still be acceptable and illustrative. The re-balancing also occurs daily, according to the strategy prescribed in Section 4.3 with the same parameter assumptions.

To benchmark the performance of perpetual American put option based strategy, a simple volatility control strategy is also applied to the same set of S&P total return scenarios. Given that the historically calibrated Heston model produces relatively volatile scenarios, the volatility target for this exercise is set to 30% to achieve the upside participation similar to the perpetual American put strategy. At each trading date, the 30-day backward realized volatility is measured and used as the volatility forecast for the next trading date. To ensure the algorithm starts with a meaningful backward volatility measurement, a header of 29 historical daily returns is concatenated in front of the simulated scenarios. The portfolio is rebalanced every trading day per the strategy prescribed in Section 4.2.

A set of 10,000 S&P total return paths are generated by Heston simulation. The perpetual American put and simple volatility control strategies produce 10,000 paths for two protected assets respectively, and therefore produce three sets of scenarios,

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

namely, the base asset, the perpetual American put strategy protected asset and simple volatility targeting strategy protected asset. Assume 70%-strike (typical level of coverage in the retail business), 5-year vanilla put options are written with each of these three assets as underlying. The put options are not dynamically hedged for the reason described at the beginning of this section. Instead, they are all hedged by a static strategy of purchasing a certain amount of vanilla put option whose underlying is the base asset and the strikes at the choice of hedger. For each scenario, the put obligation (from the issued put option) will create a claim at year 5 if the protected asset's value falls below \$70. In the meantime, the hedging asset (the purchased put option) may generate a profit depending on the choice of strike and coverage percentage.

Table 4.2: Illustrating the hedge strategy simulation

Base	Hedge Strike	%Hedged	Hedge Gain	Protected	Guarantee	Obligation	Shortfall
1323.75	80	85%	0.00	1135.94	70	0.00	0.00
79.84	80	85%	0.14	68.09	70	1.91	1.77
18.01	80	85%	52.69	80.15	70	0.00	-52.69

Table 4.2 presents three particular scenarios to further illustrate the simulation-based methodology. The simulated ending values of the base asset and protected asset at the end of the fifth year are presented in column one and five. The second and third columns specify the vanilla put hedging instruments, and for this partic-

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

ular example, one unit of the \$100 obligation is hedged by 0.8 units of put option with strike \$90. By the base scenario and the hedging instruments, the hedge gains can be obtained for each of the scenarios, which are displayed in column four. The sixth column, obligation for underwriting (in this example 70% strike) the protected asset, can be calculated from the protected asset scenarios (column five) in the way of calculating vanilla put pay-offs. The last column takes the obligation out of the hedge gain, and therefore computes the hedging shortfall. Note that once the base scenarios and the protection strategy are given, the protected asset scenarios and the obligation scenarios are also fixed. If the hedge strategy is further specified by identifying the hedge strike and hedge percentage, all seven columns in this table are fixed and the shortfall can be summarized by any meaningful statistics. This simple demonstration concentrates on the expected shortfall, and if this calculation repeats over all 10,000 scenarios, the statistics of the shortfall, including its expectation, can be easily calculated. Therefore it is reasonable to summarize this procedure as identifying the impact of hedge strike and hedge percentage on the expected hedging shortfall.

One interesting aspect to explore is the possible specifications of hedge strike and hedge percentage under a given shortfall goal, because the strike and percentage

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

directly measure the cost of hedging a guarantee written on the protected asset. As discussed at the beginning of this section, a successful protection strategy should lead to a significant reduction of hedging cost when managing a guarantee written on the protected asset. Therefore the simulation methodology above paves the way for quantifying the effectiveness of the protection strategies. Assume the hedge target is zero expected shortfall, for a given set of scenarios with both base asset and protected asset paths available, a required hedge strike can be determined with the simulation methodology described previously for every proposed hedge percentage, and vice versa. It is practical to propose a series of hedge percentages and recursively evaluate the desired hedge strike to meet the hedge target. This procedure creates a series of strike-percentage pairs which depicts the hedge costs curve in a way similar to Markowitz's portfolio management strategy. The "hedge cost frontier" introduced earlier this section helps quantifying the effectiveness of the strategies, and the hedge frontiers for hedging a \$70 guarantee of \$100 5-year investments in different protected funds are given in Figure 4.5.

Recall that the upper-right corner, where the hedge percentage is 100% and hedge strike is full, corresponds to the hedge cost of managing a guarantee on non-protected assets. The lower-left corner, where the hedge percentage and hedge strike are both

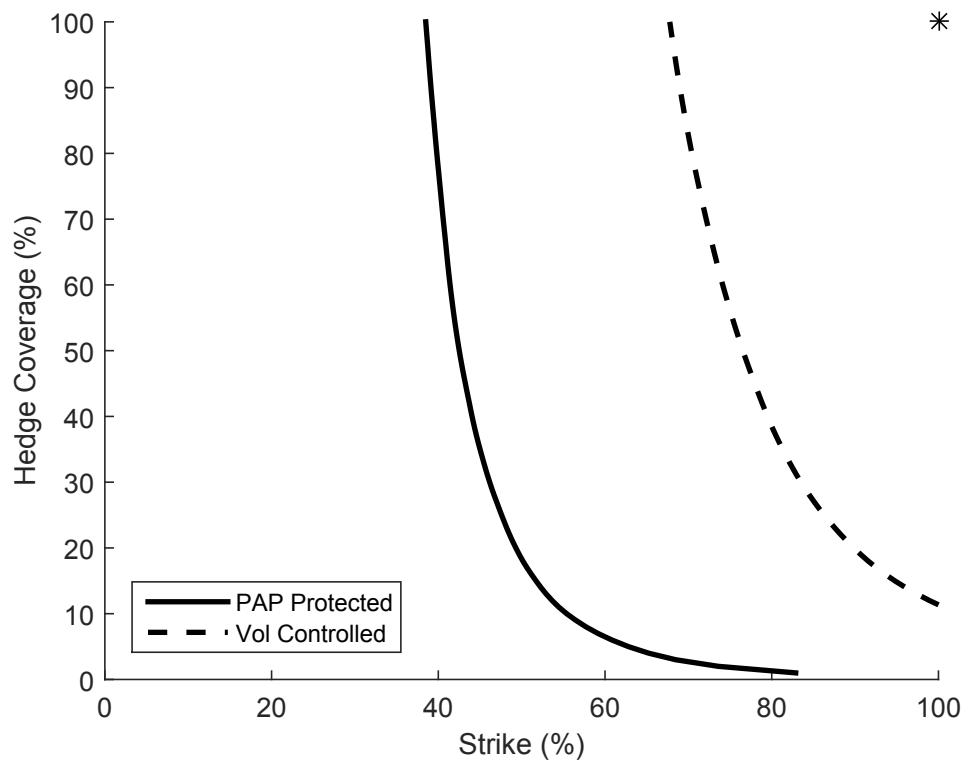


Figure 4.5: The hedge cost frontiers for both PAP (solid) and volatility targeting (dashed). Each point on the frontier indicates that to achieve the desired hedging goal, a put purchase with the combination of fractional strike ($\frac{\text{Put Strike}}{\text{Initial Value}}$) and fractional shares ($\frac{\text{Share of Puts Purchased}}{\text{Shares of Investments}}$) is required.

zero, corresponds to the hedge cost of managing a trivial guarantee written on a fully protected asset where no actual hedging is needed. The hedge cost of any other strategies will stand between these two extremes and creates a frontier shape in the box where strike is between zero and full and the coverage is between zero and 100%. A more effective protection strategy will push this frontier closer to the lower-left corner, which corresponds to lower hedge costs on the guarantee. As shown

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

in Figure 4.5, the solid curve, which outlines the hedge cost frontier of perpetual American put protection strategy, is significantly closer to the lower-left corner than the dashed curve, which is produced by the simple volatility targeting. This exercise demonstrates how hedgers are benefited from selling guarantees written on a better protected asset, given the protection strategy is fully transparent, systematic and therefore predictable.

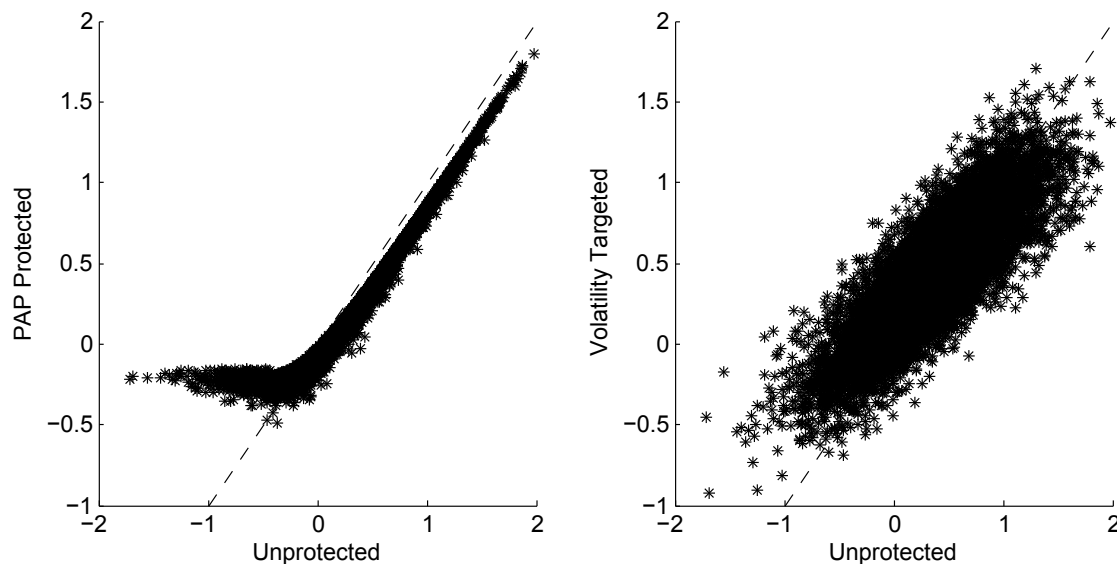


Figure 4.6: Relation of Log returns between unprotected and PAP protected funds at the end of insurance policy over all scenarios

Some other benefits of the perpetual American put strategy can be illustrated by a few more charts, even though they are not directly quantifiable. Figure 4.6 compares returns of both perpetual American put protection and simple volatility

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

targeting to the unprotected returns. The first observation is that perpetual American put protection strategy does provide very strong downside protection in that the left tail of the scatter tilts up significantly, meaning that however the unprotected asset is losing value, the mandatory withdraw rule of perpetual American put strategy prevents extreme losses.

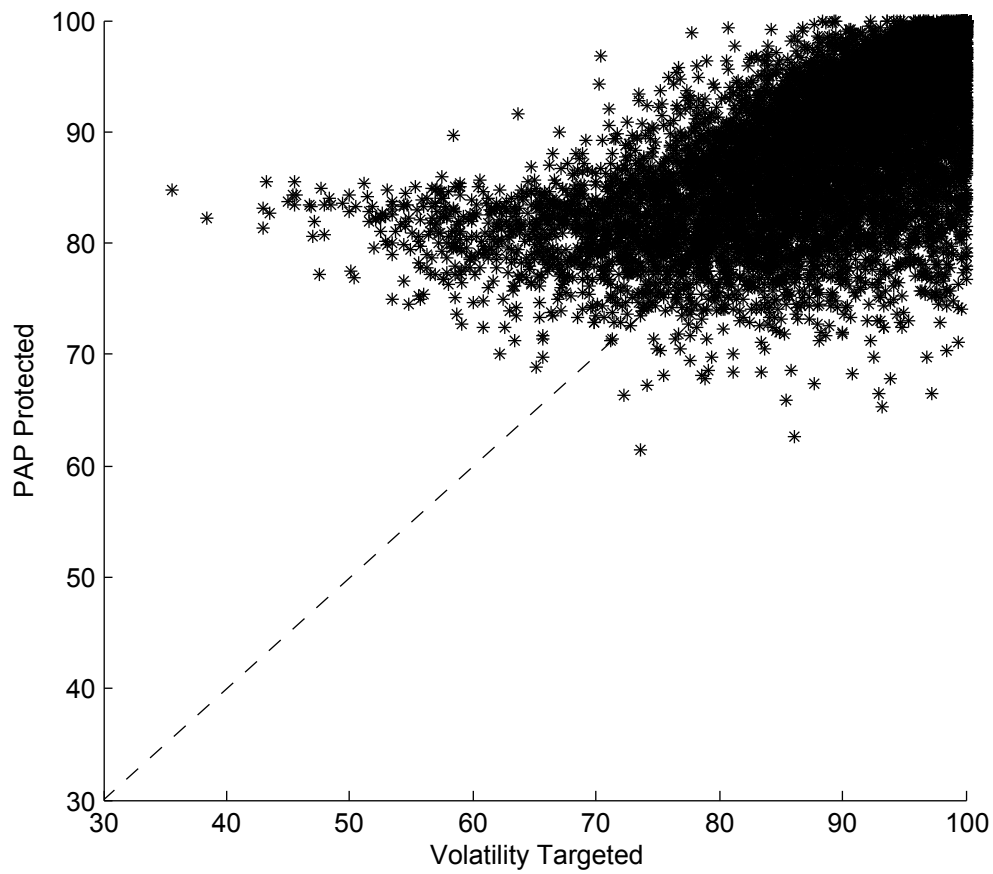


Figure 4.7: The relation of maximum loss of two strategies over all scenarios

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

A comparison between the return at maturity of two protection strategies shows that volatility targeting is much more path dependent than perpetual American put strategy, because for an identical ending base asset value, the volatility targeting strategy produces results spreading over a large interval, whereas the perpetual American put strategy's results are more condensed. The path dependency is harmful to hedgers because their hedging strategies are usually simulation-based due to the complexity of the products they are handling. High path dependency means many more paths should be generated and evaluated to achieve a desirable convergence, which heavily increases operation costs for hedging.

Finally, many of the retail-level investment insurance products provides policy holders some degree of earlier exercise privilege (either Bermuda or even American), therefore hedgers are not only concerned about the ending value of the protected asset, but also its maximum losses throughout the insurance policy period. Figure 4.7 compares two protection strategies by the path-wise minimum value of the protected asset. It can be seen that the perpetual American put strategy's mandatory withdraw rule covers not only the maturity of the product but also throughout the entire course of investment. When the simple volatility targeting strategy suffers from significant drawdowns, the perpetual American put strategy prevents the protected asset from

CHAPTER 4. APPLICATION IN INVESTMENT OVERLAY

falling so low that an early exercise of the investment insurance becomes retail customers' best interests. Even though volatility targeting shows better performance in small drawdown scenarios, it is not particularly helpful because this kind of mild decline is unlikely to trigger early exercises/claims from insurance policy holders. Also, because of the high premium charged upfront, hedgers can usually withstand small drawdowns on a deductible basis by accepting a small percentage (e.g. 10%) of initial losses. A stronger protection to large drawdowns provided by perpetual American put strategy is more desirable.

Chapter 5

Conclusions and Future Work

Section 5.1 reviews and summarizes the outcomes of this dissertation. Section 5.2 outlines areas where the perpetual American put CEV structural model can be refined or extended to become more applicable to equity options market.

5.1 Conclusions

Structural modeling of leveraged firms reveals additional properties of the firm's equity stochastic. The model presented in this dissertation assumes a constant perpetual nominal liability and a CEV dynamic on the asset of the firm. The latter helps to achieve a balance between a realistic asset dynamic and a mathematically tractable model. This structural model has a better potential for calibrating to the

CHAPTER 5. CONCLUSIONS AND FUTURE WORK

volatility structures observed in the market, thanks to its additional degree of freedom. It also agrees well with previous results presented by Merton and Leland.

Through the adoption of the CEV process, the capital structure of a limited liability firm can be characterized by a free boundary differential equation, which leads to a highly desirable analytical relation between equity, assets, and liabilities. Such an analytical relation is the key to simplifying the equity option from a compound option into a more approachable barrier option, and therefore eventually enabling the calibration of model parameters to market prices of vanilla options. This calibration eliminates the need for judgmental data processing of financial statements, as seen in many previous structural implementations, and consequently provides a more objective assessment of the firm's fundamentals.

As a byproduct, the decomposition of a firm's equity inspired an investment protection strategy applicable to retail fund management. This path-insensitive investment overlay provides strong protection against heavy market loss and at the same time is fully systematic and predictable. Such features are highly desirable to insurance providers such as variable annuity hedgers.

5.2 Future Work

This dissertation has focused on integrating several important concepts to propose an applicable framework that can fit structural models into the stock options market. However, similar to what is seen in Leland (1994) and Toft and Prucyk (1997), the martingale property is lost when constructing a stochastic equity model under a perpetual framework. This loss prevents the perpetual structural model from serving as a market pricing model or hedging model.

One possible refinement could be alternating the nominal liability specification, so that the nominal liability term in the equity decomposition can become a discounted martingale as well. However, this heavily complicates the pricing of the perpetual American put option. The research on stock loan pricing by Xia and Zhou (2007) provides a good starting point for pricing a perpetual put option with a time-varying strike. However, this is achieved under the GBM asset dynamic specification, and the generalization to CEV asset assumption is not yet known. As observed in Chapter 3, from time to time, an asymmetric asset dynamic is necessary in order to fit the structural model to volatility skews and smiles. The analytical pricing of a perpetual put option with a time-varying strike under the CEV framework would make a significant contribution to structural modeling.

CHAPTER 5. CONCLUSIONS AND FUTURE WORK

When a structural model retains the martingale property, it is also a valid pricing and hedging model for the equity option market. Taking Merton's model as an example, the volatility function of the equity dynamic is both time and level dependent, and therefore naturally creates a local volatility model that is desirable to equity derivative traders and hedgers. It has been a challenge to construct a local volatility function that creates an arbitrage-free specification of the volatility surface. In the case of a structural model, because the equity option is constructed as a compound option (or option of an option), the arbitrage-free requirement is naturally satisfied. Therefore Merton's model has the potential to provide a simple yet effective specification of arbitrage-free volatility surfaces. The challenge is that usually there are richer volatility term structures expectations in the market, so the constant asset volatility assumption in Merton's model needs to be refined to incorporate such a non-constant market opinion.

In summary, this dissertation demonstrates the potential of structural leverage model in various aspects besides the modeling of bankruptcy and default. We will continue to devote efforts to this topic and expect to find more exciting results.

Appendix A

Notations

α : the CEV elasticity

B_t : a zero coupon bond maturing at time t with pay-off of \$1

C_K : the price of stock option with strike K

\tilde{D} : the firm's nominal liability value

D or $D(V)$: the firm's market value of liability

Δ : the absolute Delta of the put option in the context of fund management

$E(V)$ or $E(V(t))$ or $E(t)$ or E : the stochastic process of the firm's equity value

$\tilde{\mathbb{E}}$: the \mathbb{Q} -measure expectation

$\mathcal{F}, \mathcal{F}_t$: the information filtration and its σ -algebra at time t

ϕ : the CEV volatility

APPENDIX A. NOTATIONS

G : the strike of perpetual option in fund management strategy

$H < \cdot, \cdot >$: the Hellinger distance

$\mathbb{I}_{\{\cdot\}}$: the indicator function

K : the strike of stock option in the context of leveraged firm

$\kappa_0, \kappa_1, \kappa_2, \kappa_3$: the integration constants to be determined by boundary conditions in the general solution of $E(V)$

L : the default level in the context of leveraged firm, or the mandatory pull-out value in the context of fund management

PD_T : probability of default by time T

$\tilde{\mathbb{P}}$: the \mathbb{Q} -measure probability

\mathbb{Q} : the risk-neutral probability measure

r : the risk-free interest rate

t : time

$V(t)$ or V : the stochastic process of the firm's asset value

\tilde{W} : the \mathbb{Q} measure Brownian motion

Appendix B

Additional Literatures

Black and Cox (1976) provided an intensive study on bond indenture provisions including safety covenants, subordinations and interest/dividend restrictions. Geske (1977) priced risky coupon bond as a compound option. Myers (1977) explored the inverse relation between corporate borrowing and market value under the structural framework. Brennan and Schwartz (1978) proposed a structural model with taxes under consideration. Geske (1979) has been reviewed in detail in Section 1.2.

Jones et al. (1984) has also been reviewed in detail in Section 1.2. Brennan and Schwartz (1984) explored structural modeling with both corporate asset and asset return as state variables and allowed firms to make choices on both investment and

APPENDIX B. ADDITIONAL LITERATURES

financing policies. Fischer et al. (1989) explored the “the firm’s optimal restructuring” of capital structures “in response to fluctuations in asset values over time”. Mello and Parsons (1992) extended structural model by introducing incentive effect and agency costs. Kim et al. (1993) studied the yield spread of corporate bonds by applying contingent claim approach to both corporate and treasury bonds. The innovative work Leland (1994) on perpetual option based structural modeling has been reviewed in detail in Section 1.2.

Longstaff and Schwartz (1995) provided a framework to study the interaction between default and interest rate risk. Anderson and Sundaresan (1996) provided a study combining structural model with debt designs. At the same year, Leland and Toft (1996) extended Leland (1994) model to accommodate the study of optimal debt maturity and amount. Briysa and de Varenne (1997) extended finite maturity default-able bond valuation to include early default and interest rate risk. Fries et al. (1997) studied optimal leverage level by incorporating price elasticity into debt valuation. Jarrow et al. (1997) modeled bankruptcy process with discrete Markov chain. Mella-Barral and Perraudin (1997) studied the strategic service of debt, which allowed for concessions from debtholders to reduce the contracted liability obligations. Ross (1997) introduced optimal selection of volatility and dividend policies into the

APPENDIX B. ADDITIONAL LITERATURES

framework of Leland and Toft (1996)'s. Toft and Prucyk (1997) for the first time made a direct connection between leveraged structural model and volatility skew. This paper has been reviewed in detail in Section 1.2.

Leland (1998) found that “agency costs restrict leverage and debt maturity and increase yield spreads”. Madan and Unal (1998) achieved a decomposition of default risk into timing and recovery risks. Duffee (1999) modeled probability of default with a translated square-root diffusion process to allow for interest rate correlation and achieved successful fitting to corporate bond yields. Duffie and Singleton (1999) proposed a reduced-form model where the default is assumed to be unpredictable hazard-rate process. Mella-Barral (1999) extended the Mella-Barral and Perraudin (1997) work to model the bargaining process which prevents the premature or deferred decision of default. Anderson and Sundaresan (2000) revealed the advantage of endogenous bankruptcy verses original Merton (1974) model. Ericsson (2000) provided further analysis on leverage, maturity and agency costs. Fan and Sundaresan (2000) provided a comprehensive study of equity/debt valuation, reorganization and optimal dividends.

APPENDIX B. ADDITIONAL LITERATURES

Collin-Dufresne and Goldstein (2001) presented a structural model with adjustable leverage. Duffie and Lando (2001) explored structural modeling under imperfect information to the asset stochastic. Goldstein et al. (2001) proposed the first systematic structural framework with dynamic capital structure. Gordy and Heitfield (2001) presented a rating assignment model based on structural approach. Morellec (2001) studied the impact of asset liquidity on leverage and credit spread. Zhou (2001) incorporates jump risk into the structural modeling and introduced pairwise correlation to structural modeling.

Acharya and Carpenter (2002) introduced stochastic interest rates into default modeling. Hilberink and Rogers (2002) extended Leland and Toft (1996) work by introducing downward jumps and resorting to numerical solutions. Brockman and Turtle (2003) modeled the equity of a firm as a down-and-out European call. Delianedis and Geske (2003) studied the risk neutral default probabilities under both Merton (1974) and Geske (1977) frameworks, and reported these probabilities are effective measures to firms' credit qualities. Huang and Huang (2003) found that credit yield accounts only for a small proportion of corporate bond spread by calibrating a wide variety of structural models. Sarkar and Zapatero (2003) proposed a structural model based on trade-off theory, perpetual option framework and mean reverting asset pro-

APPENDIX B. ADDITIONAL LITERATURES

cess.

Dangl and Zechner (2004) explored the optimal leverage under the structural framework. Y. H. Eom (2004) provided an empirical study of the predictive power of several structural models on corporate bond spread. Ericsson and Reneby (2004) showed that it is unnecessary to assume tradability of the firm's underlying asset value when at least one other of firm's asset (e.g. equity or bond) is traded, thus provides a strong theoretical validation to the structural modeling approach. Francois and Morellec (2004) introduced debt renegotiation into structural modeling. Giesecke and Goldberg (2004) integrated the structural model with the reduced-form model. Hull et al. (2004a) has been reviewed in detail in Section 1.2. N. Ju et al. (2005) explored the optimal capital structure based on tax shields, bankruptcy costs and total firm value. Morellec (2004) integrated the “managerial discretion and corporate control mechanisms” into the structural modeling framework. Vasalou and Xing (2004) proposed to calibrate Merton's model to historical realized equity volatilities.

Brigo and Tarenghi (2005) demonstrated a successful CDS calibration with a structural model based on tractable barrier option. Childs et al. (2005) provided a

APPENDIX B. ADDITIONAL LITERATURES

two dimension structural model which includes “asset-in-place” and “asset underlying the growth option”. Elizalde (2006) provided a thorough review of existing structural models. Ericsson and Reneby (2005) proposed a maximum likelihood alternative to the estimation of the unobservable asset value of the firm. Arora et al. (2005) provided a comparative study between structural and reduced-form models. Ericsson and Renault (2006) developed “a structural bond valuation model to simultaneously capture liquidity and credit risk”. Hackbarth et al. (2006) proposed a macroeconomic structural model “in which the firms’ cash flows depend on both an idiosyncratic shock and an aggregate shock that reflects the state of economy.” Ju and H.O.Yang (2006) studied the capital structure based on a structural model which integrates the Vasicek stochastic interest rate. Leland (2006) also provided a thorough review of the existing structural models.

Ahangarani (2007) proposed a new structural model based on the theory that equity price is determined by expected dividends. Broadie et al. (2007) studied optimal capital structure based on structural model with interests of borrowers and lenders in consideration. Davydenko and Strebulaev (2007) studied the impact from strategic actions of borrowers and lenders based on a model similar to Leland (1994). Décamps and Djembissi (2007) provided a study on asset substitution under the

APPENDIX B. ADDITIONAL LITERATURES

structural modeling framework. Ericsson et al. (2007) provided an empirical study of Merton (1974), Leland and Toft (1996) and Fan and Sundaresan (2000) models in terms of explaining the CDS spreads. Hackbarth et al. (2007) provided another study of structural model with trade-off theory. Schäfer et al. (2007) introduces jump and default correlations into structural modeling and applied to a credit portfolio. Strebulaev (2007) studied the effectiveness of structural trade-off models by statistical tests. Titman and Tsyplakov (2007) presented a model with both dynamic capital structure and investment choices.

Albuquerque and Wang (2008) developed a structural model to address the “separation of ownership and control” of firms. Bharath and Shumway (2008) reported the Merton (1974) model is underperforming a reduced-form alternative. Hackbarth (2008) incorporates the managerial traits into a trade-off structural model. Lambrecht and Myers (2008) presented a structural model considering managers’ future compensation. Schaefer and Strebulaev (2008) showed that structural model provides good prediction to the relation between corporate bond prices and the equity value. Tserlukevich (2008) provided a trade-off structural model with no transaction cost but with investment costs. Wong and Choi (2009) improved Brockman and Turtle (2003) work by proposing a statistical estimation of default barriers.

APPENDIX B. ADDITIONAL LITERATURES

Chen and Kou (2009) has been reviewed in detail in Section 1.2. Forte (2011) improved Leland and Toft (1996) by carrying out a joint calibration to equity and CDS. Bhamra et al. (2010) embedded a consumption-based asset-pricing model into structural framework to address the impact of macroeconomic conditions. Dorfleitner et al. (2011) proposed a model based on non-constant barrier option which relies on numerical solution. Hull et al. (2010) extended Zhou (2001) work to address the correlation between a large population of firms. Morellec and Schürhoff (2010) studied “the effects of capital gains taxation on firms’ investment and financing decisions”. Pennacchi (2010) applied the structural modeling in the area of bank capital. Shibata and Nishihara (2010) provided another study on manager-shareholder conflict. Lee (2011) introduced genetic algorithms to solve the optimal default boundary in KMV’s model. Morellec and Schürhoff (2011) integrated the corporate managements’ information advantage into structural modeling.

Some of these papers belong to the class of exogenous default modeling, where the default is triggered when asset value crosses a critical value preset by the modeler, including Longstaff and Schwartz (1995), Kim et al. (1993), Briysa and de Varenne (1997), Collin-Dufresne and Goldstein (2001). Brennan and Schwartz (1980) is an-

APPENDIX B. ADDITIONAL LITERATURES

other noticeable one.

Appendix C

Historical Maximum Likelihood

Estimation of Heston Model

The Heston stochastic volatility equity model considered in this thesis can be characterized by the following SDEs:

$$\begin{cases} dS = \mu S dt + \sqrt{\nu} S dW^1 \\ d\nu = \kappa(\theta - \nu) dt + \xi \sqrt{\nu} dW^2 \end{cases} \quad (\text{C.1})$$

where the stock price S follows a log-normal process with drift μ and stochastic volatility $\sqrt{\nu}$. ν follows a square-root process governed by mean reversion factor κ , equilibrium variance level θ and diffusion factor ξ . The two Brownian motions are

APPENDIX C. HISTORICAL MAXIMUM LIKELIHOOD ESTIMATION OF HESTON MODEL

correlated so that $dW^1 dW^2 = \rho dt$.

Under such a specification, the incrementals, dS and $d\nu$ follow bivariate Gaussian distribution:

$$\begin{bmatrix} ds \\ d\nu \end{bmatrix} \sim N \left(\begin{bmatrix} \mu S dt \\ \kappa(\theta - \nu) dt \end{bmatrix}, \begin{bmatrix} \nu S^2 dt & \rho \nu \xi S dt \\ \rho \nu \xi S dt & \nu \xi^2 dt \end{bmatrix} \right) \quad (\text{C.2})$$

Given a history of price and variance incrementals, $\{dS_1, dS_2, \dots, dS_n\}$ and $\{d\nu_1, d\nu_2, \dots, d\nu_n\}$, over small time step dt , maximizing the log-likelihood will obtain a historical calibration of all five Heston parameters ($\{\mu, \kappa, \theta, \xi, \rho\}$, denoted by α):

$$\tilde{\alpha} = \operatorname{argmax}_{\alpha} \sum_{i=1}^n \ln f(dS_i, d\nu_i | \alpha) \quad (\text{C.3})$$

where f is the joint density function of the bivariate normal distribution defined by (C.2).

In reality, the return variance, or equivalently the stock price volatility, is a latent variable that cannot be easily observed. Several approximation approaches exist. S&P index, for example, has its volatility being traded through VIX index futures, so that the VIX level can serve as a good approximation to the S&P volatility. More

APPENDIX C. HISTORICAL MAXIMUM LIKELIHOOD ESTIMATION OF HESTON MODEL

generally, other volatility measures can be used to extract variance sequence from the price history. In this thesis, the price history is first converted into return history and then fed into GARCH(1,1) model for volatility/variance estimation.

Bibliography

- Acharya, V. V. and Carpenter, J. N. (2002). Corporate bond valuation and hedging with stochastic interest rates and endogenous bankruptcy. *Review of Financial Studies*, 15(15):1355–1383.
- Ahangarani, P. M. (2007). A new structural approach to the default risk of companies. *Working paper*.
- Albanese, C. and Campolieti, G. (2006). *Advanced Derivatives Pricing and Risk Management: Theory, Tools and Hands-on Programming Application*. Academic Press.
- Albuquerque, R. and Wang, N. (2008). Agency conflicts, investment and asset pricing. *The Journal of Finance*, 63(1):1–40.
- Anderson, R. and Sundaresan, S. (2000). A comparative study of structural models

BIBLIOGRAPHY

- of corporate bond yields: An exploratory investigation. *Journal of Banking & Finance*, 24(1-2):255–269.
- Anderson, R. W. and Sundaresan, S. (1996). Design and valuation of debt contracts. *Review of Financial Studies*, 9(1):37–68.
- Arora, N., Bohn, J. R., and Zhu, F. (2005). Reduced-form vs. structural models of credit risk. *Journal of Investment Management*, 3(4).
- Bhamra, H., Kuehn, L., and Strebulaev, I. A. (2010). The aggregate dynamics of capital structure and macroeconomic risk. *Review of Financial Studies*, 23(12):4187–4241.
- Bharath, S. T. and Shumway, T. (2008). Forecasting default with the merton distance to default model. *Review of Financial Studies*, 21(3):1339–1369.
- Black, F. and Cox, J. C. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367.
- Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.
- Brennan, M. and Schwartz, E. (1980). Analyzing convertible bonds. *The Journal of Financial and Quantitative Analysis*, 15(4):907–929.

BIBLIOGRAPHY

- Brennan, M. J. and Schwartz, E. S. (1978). Corporate income taxes, valuation, and the problem of optimal capital structure. *The Journal of Business*, 51(1):103–114.
- Brennan, M. J. and Schwartz, E. S. (1984). Optimal financial policy and firm valuation. *The Journal of Finance*, 39(3):593–607.
- Brigo, D. and Tarengi, M. (2005). Credit default swap calibration and equity swap valuation under counterparty risk with a tractable structural model. *SSRN working paper*.
- Briysa, E. and de Varenne, F. (1997). Valuing risky fixed rate debt: An extension. *Journal of Financial and Quantitative Analysis*, 32(2):239–248.
- Broadie, M., Chernov, M., and Sundaresan, S. (2007). Capital structure, debt maturity, and stochastic interest rates. *The journal of Finance*, 62(3):1341–1377.
- Brockman, P. and Turtle, H. (2003). A barrier option framework for corporate security valuation. *Journal of Financial Economics*, 67(3):511–529.
- Carr, P., Geman, H., Madan, D. B., and Yor, M. (2002). The fine structure of asset returns: An empirical investigation. *The Journal of Business*, 75-2:305–332.
- Chen, N. and Kou, S. G. (2009). Credit spreads, optimal capital structure, and

BIBLIOGRAPHY

- implied volatility with endogenous default and jump risk. *Mathematical Finance*, 19:343–378.
- Childs, P., Mauer, D. C., and Ott, S. H. (2005). Interactions of corporate financing and investment decisions: The effects of agency conflicts. *Journal of financial economics*, 76(3):667–690.
- Collin-Dufresne, P. and Goldstein, R. S. (2001). Do credit spreads reflect stationary leverage ratios? *The Journal of Finance*, 56(5):1929–1957.
- Cox, J. C. (1996). The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, Special Issue:15–17.
- Crouhy, M., Galai, D., and Mark, R. (2000). A comparative analysis of current credit risk models. *Journal of Banking & Finance*, 24:59–117.
- Dangl, T. and Zechner, J. (2004). Credit risk and dynamic capital structure choice. *Journal of Financial Intermediation*, 13(2):183–204.
- Davydenko, S. A. and Strebulaev, I. A. (2007). Strategic actions and credit spreads: An empirical investigation. *The journal of Finance*, 62(6):2633–2671.
- Davydov, D. and Linetsky, V. (2001). Pricing and hedging path-dependent options under the cev process. *Management Science*, 47(7):949–965.

BIBLIOGRAPHY

- Décamps, J. P. and Djembissi, B. (2007). Switching to a poor business activity: Optimal capital structure, agency costs and covenant rules. *Annals of Finance*, 3(3):389–409.
- Delianedis, G. and Geske, R. (2003). Credit risk and risk neutral default probabilities: Information about rating migrations and defaults. *EFA 2003 annual conference paper*, 962:511–529.
- Dorflleitner, G., Schneider, P., and Veža, T. (2011). Flexing the default barrier. *Quantitative Finance*, 11(12):1729–1743.
- Duffee, G. R. (1999). Estimating the price of default risk. *Review of Financial Studies*, 12(1):197–226.
- Duffie, D. and Lando, D. (2001). Term structures of credit spreads with incomplete accounting information. *Econometrica*, 69(3):633–664.
- Duffie, D. and Singleton, K. J. (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12(4):687–720.
- Dupire, B. (1994). Pricing with a smile. *Risk*, February:18–20.
- Ekström, E. (2003). Perpetual American put options in a level-dependent volatility model. *Journal of Applied Probabilities*, 40:783–789.

BIBLIOGRAPHY

- Elizalde, A. (2006). Credit risk models ii: Structural models. *Documentos de Trabajo (CEMFI)s*, 6:1.
- Ericsson, J. (2000). Asset substitution, debt pricing, optimal leverage and maturity. *Working paper*.
- Ericsson, J. and Renault, O. (2006). Liquidity and credit risk. *The Journal of Finance*, 61(5):2219–2250.
- Ericsson, J. and Reneby, J. (2004). A note on contingent claims pricing with non-traded assets. *Working paper*.
- Ericsson, J. and Reneby, J. (2005). Estimating structural bond pricing models. *Journal of Business*, 78(2):707–735.
- Ericsson, J., Reneby, J., and Wang, H. (2007). Can structural models price default risk? new evidence from bond and credit derivative markets. *SSRN working paper*.
- Fan, H. and Sundaresan, S. M. (2000). Debt valuation, renegotiation, and optimal dividend policy. *Review of Financial Studies*, 13(4):1057–1099.
- Fischer, E. O., Heinkel, R., and Zechner, J. (1989). Dynamic capital structure choice: Theory and tests. *The Journal of Finance*, 44(1):19–40.

BIBLIOGRAPHY

- Forte, S. (2011). Calibrating structural models: A new methodology based on stock and credit default swap data. *Quantitative Finance*, 11(12):1745–1759.
- Francois, P. and Morellec, E. (2004). Capital structure and asset prices: Some effects of bankruptcy procedures. *The Journal of Business*, 77(2):387–411.
- Fries, S., Miller, M., and Perraudin, W. (1997). Debt in industry equilibrium. *Review of Financial Studies*, 10(1):39–67.
- Geske, R. (1977). The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis*, 12(04):541–552.
- Geske, R. (1979). The valuation of compound options. *Journal of Financial Economics*, 7:63–81.
- Giesecke, K. and Goldberg, L. (2004). Sequential defaults and incomplete information. *Journal of Risk*, 7:1–26.
- Goldstein, R., Ju, N., and Leland, H. (2001). An ebit - based model of dynamic capital structure. *The Journal of Business*, 74(4):483–512.
- Gordy, M. and Heitfield, E. (2001). Of moody’s and merton: a structural model of bond rating transitions. *Working Paper*.

BIBLIOGRAPHY

- Hackbarth, D. (2008). Managerial traits and capital structure decisions. *Journal of Financial and Quantitative Analysis*, 43(04):843–881.
- Hackbarth, D., Hennessy, C. A., and Leland, H. E. (2007). Can the trade-off theory explain debt structure? *Review of Financial Studies*, 20(5):1389–1428.
- Hackbarth, D., Miao, J., and Morellec, E. (2006). Capital structure, credit risk, and macroeconomic conditions. *Journal of Financial Economics*, 82(3):519–550.
- Heston, S. (1993). A closed-form solution of options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6:327–343.
- Hilberink, B. and Rogers, L. (2002). Corporate bond valuation and hedging with stochastic interest rates and endogenous bankruptcy. *Finance and Stochastics*, 6(2):237–263.
- Hsu, Y. L., Lin, T. I., and Lee, C. F. (2010). Constant elasticity of variance option pricing model, integration and detailed derivation. *Handbook of Quantitative Finance and Risk Management*, III:471–480.
- Huang, J. and Huang, M. (2003). How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies*, 2(2):153–202.

BIBLIOGRAPHY

- Hull, J., Predescu, M., and White, A. (2010). The valuation of correlation-dependent credit derivatives using a structural model. *SSRN working paper*.
- Hull, J. C., Nelken, I., and White, A. D. (2004a). Merton's model, credit risk and volatility skews. *Journal of Credit Risk*, 1-1:3–27.
- Hull, J. C., Predescu, M., and White, A. D. (2004b). The relationship between credit default swap spreads, bond yields, and credit rating announcements. *Journal of Banking and Finance*, 28:2789–2811.
- Ingersoll, J. E. (1987). *Theory of Financial Decision Making*. Rowman & Littlefield Publishers.
- Jarrow, R. A., Lando, D., and Turnbull, S. M. (1997). A markov model for the term structure of credit risk spreads. *Review of Financial Studies*, 10(2):481–523.
- Jones, E. P., Mason, S. P., and Rosenfeld, E. (1984). Contingent claims analysis of corporate capital structure: an empirical investigation. *Journal of Finance*, 39:611–625.
- Ju, N. and H.O. Yang (2006). Capital structure, debt maturity, and stochastic interest rates. *The Journal of Business*, 79(5):2469–2502.
- Kim, J., Ramaswamy, K., and Sundaresan, S. (1993). Does default risk in coupons

BIBLIOGRAPHY

- affect the valuation of corporate bonds?: A contingent claims model. *Financial Management*, 22(3):117–131.
- Lambrecht, B. M. and Myers, S. C. (2008). Debt and managerial rents in a real-options model of the firm. *Journal of Financial Economics*, 89(2):209–231.
- Lee, W. (2011). Redefinition of the kmv model’s optimal default point based on genetic algorithms - evidence from taiwan. *Expert Systems with Applicationse*, 38(8):10107–10113.
- Leland, H. (1994). Debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49:1213–1252.
- Leland, H. (1998). Agency costs, risk management, and capital structure. *The Journal of Finance*, 53(4):1213–1243.
- Leland, H. E. (2006). *Predictions of Default Probabilities in Structural Models of debt*. Wiley.
- Leland, H. E. and Toft, K. B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019.

BIBLIOGRAPHY

- Longstaff, F. A. and Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819.
- Madan, D. B. and Unal, H. (1998). Pricing the risks of default. *Review of Derivatives Research*, 2(2-3):121–160.
- Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.
- Mella-Barral, P. (1999). The dynamics of default and debt reorganization. *Review of Financial Studies*, 12(3):535–578.
- Mella-Barral, P. and Perraudin, W. (1997). Strategic debt service. *The Journal of Finance*, 52(2):531–556.
- Mello, A. S. and Parsons, J. E. (1992). Measuring the agency cost of debt. *The Journal of Finance*, 47(5):1887–1904.
- Merton, R. C. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29:449–470.
- Merton, R. C. (1976). Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics*, 4:125–144.
- Morellec, E. (2001). Asset liquidity, capital structure and secured debt. *Journal of Financial Economics*, 61(2):173–206.

BIBLIOGRAPHY

- Morellec, E. (2004). Can managerial discretion explain observed leverage ratios? *Review of Financial Studies*, 17(1):257–294.
- Morellec, E. and Schürhoff, N. (2010). Dynamic investment and financing under personal taxation. *Review of Financial Studies*, 23(1):101–146.
- Morellec, E. and Schürhoff, N. (2011). Dynamic investment and financing under asymmetric information. *Working paper*.
- Moskowitz, B. and Caffisch, R. E. (1996). Smoothness and dimension reduction in quasi-monte carlo methods. *Mathematical and Computer Modelling*, 23(8-9):37–54.
- Myers, S. C. (1977). Determinants of corporate borrowing. *Journal of Financial Economics*, 5(2):147–175.
- N. Ju, R. P., Poteshman, A., and Weisbach, M. (2005). Horses and rabbits? trade-off theory and optimal capital structure. *Journal of Financial and Quantitative Analysis*, 40(02):259–281.
- Pennacchi, G. (2010). A structural model of contingent bank capital. *SSRN working paper*.
- Ross, M. P. (1997). Dynamic optimal risk management and dividend policy under optimal capital sturcutre and maturity. *Working paper*.

BIBLIOGRAPHY

- Samuelson, P. A. (1965). Rational theory of warrant pricing. *Industrial Management Review*, 6(2):13–39.
- Sarkar, S. and Zapatero, F. (2003). The trade-off model with mean reverting earnings: Theory and empirical tests. *The Economic Journal*, 113(490):834–860.
- Schaefer, S. M. and Strebulaev, I. (2008). Structural models of credit risk are useful: Evidence from hedge ratios on corporate bonds. *Journal of Financial Economics*, 90(1):1–19.
- Schäfer, R., Sjölin, M., Sundin, A., Wolanski, M., and Guhr, T. (2007). Credit risk - a structural model with jumps and correlations. *Physica A: Statistical Mechanics and its Applications*, 383(2):533–569.
- Shibata, T. and Nishihara, M. (2010). Dynamic investment and capital structure with manager-shareholder conflict. *Journal of Economic Dynamics and Control*, 34(2):158–178.
- Strebulaev, I. (2007). Do tests of capital structure theory mean what they say? *The Journal of Finance*, 62(4):1747–1787.
- Titman, S. and Tsyplakov, S. (2007). A dynamic model of optimal capital structure. *Review of Finance*, 11(3):401–451.

BIBLIOGRAPHY

- Toft, K. B. and Prucyk, B. (1997). Options on leveraged equity: theory and empirical tests. *Journal of Finance*, 52:1151–1180.
- Tserlukevich, Y. (2008). Can real options explain financing behavior? *Journal of Financial Economics*, 89(2):232–252.
- Vassalou, M. and Xing, Y. (2004). Default risk in equity returns. *The Journal of Finances*, 59(2):831–868.
- Wong, H. Y. and Choi, T. W. (2009). Estimating default barriers from market information. *Quantitative Finance*, 9(2):187–196.
- Xia, J. and Zhou, X. Y. (2007). Stock loans. *Mathematical Finance*, 17(2):307–317.
- Y. H. Eom, J. Helwege, J. H. (2004). Structural models of corporate bond pricing: An empirical analysis. *Review of Financial studies*, 17(2):499–544.
- Zhou, C. (2001). The term structure of credit spreads with jump risk. *Journal of Banking & Finance*, 25(11):2015–2040.

Vita



Min Chen received B.Sc. degree in Theoretical and Applied Mechanics from Fudan University, Shanghai, China in 2006, and M.Sc. degree in Applied Mathematics and Statistics from Johns Hopkins University in 2008. His research focuses on stochastic modeling of financial instruments and investment overlay strategies. From August 2013 to January 2016, Min served as a quantitative analyst in the Variable Annuity Hedging team in Ohio National Financial Services, Cincinnati, Ohio. He was also a research intern from May to July 2013 at Campbell & Company, a Baltimore based hedge fund.